A new light on Minkowski's $?(x)$ function

Pelegrí Viader[1]
Jaume Paradís[1]
and
Lluís Bibiloni[2]

June 1997

Journal of Economic Literature classification: C00

1991 *Mathematics Subject Classification.* Primary 11K31; Secondary 26A30.

Keywords and phrases. Asymptotic distribution functions, Minkowski's function, singular functions.

[1] Applied Mathematics, UPF. Ramon Trias Fargas 25–27, 08005 Barcelona, Spain.
[2] Facultat de Ciències de l'Educació, UAB. 08193 Bellaterra, Barcelona, Spain.

Abstract

The well-known Minkowski's $?(x)$ function is presented as the asymptotic distribution function of an enumeration of the rationals in $(0,1]$ based on their continued fraction representation. Besides, the singularity of $?(x)$ is clearly proved in two ways: by exhibiting a set of measure one in which $?'(x) = 0$; and again by actually finding a set of measure one which is mapped onto a set of measure zero and viceversa. These sets are described by means of metrical properties of different systems for real number representation.

1 Introduction

Minkowski's $?(x)$ function was introduced by Minkowski (see [7]) for the purpose of establishing a new criterium for quadratic irrationals based on a one-to-one correspondence between some rational numbers and the quadratic irrationals of $[0,1]$. Minkowski's original construction is very simple: on the x axis he 'draws' the rationals by means of the mediants in the Farey fractions and to each of these mediants he assigns on the y axis the corresponding dyadic division point. The function is extended to all $x \in [0,1]$ by continuity. Denjoy in [2] studied the function and proved it to be a strictly increasing singular function.

For the sake of completeness we present the definition of $?(x)$ as it is given by Salem in [15]. First we define:

$$?(0) = ?(0/1) = 0, \qquad ?(1) = ?(1/1) = 1.$$

Then we take the mediant $1/2 = (0+1)/(1+1)$ between the two Farey fractions $0/1$ and $1/1$ and we define

$$?(1/2) = \frac{?(0)+?(1)}{2} = \frac{1}{2};$$

we continue in the same way,

$$?\left(\frac{p+p'}{q+q'}\right) = \frac{?(p/q)+?(p'/q')}{2}.$$

The definition for irrational x follows by continuity.

Salem, in the same article, finds a new presentation for $?(x)$. If $x \in (0,1]$ is developed as a regular continued fraction:

$$x = [0; a_1, a_2, \ldots, a_n, \ldots],$$

then

$$?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \frac{1}{2^{a_1+a_2+a_3-1}} - \cdots \tag{1}$$

From this definition, Salem draws all the important properties of $?(x)$:

1. x is a quadratic irrational iff $?(x)$ is a rational with a non-terminating expansion.

2. $?(x)$ is strictly increasing.

3. $?(x)$ is a singular function, that is, its derivative is 0 almost everywhere (in the sense of the measure of Lebesgue).

The set found by Salem, on which the derivative of $?(x)$ is zero is the intersection of

$$N = \{x = [0; a_1, a_2, \ldots] : \limsup a_n = \infty\},$$

with the set of the points in $(0,1]$ on which $?(x)$ has a finite derivative. Both sets are of measure one. This presentation of Salem had been inadvertently introduced by Ryde in 1926 (see [14] for the details) without the connexion with Minkowski's function.

In section 2 of this paper, we present a new way of looking at $?(x)$, by obtaining it as the asymptotic distribution funtion (a.d.f.) of a sequence. A function $F(x)$ is called the a.d.f. of the sequence $\{q(n)\}, 0 \leq q(n) \leq 1$ if:

$$\lim_{n\to\infty} \frac{\#\{q(i) \leq x\,;\ i = 1, 2, \ldots, n\}}{n} = F(x) \qquad \text{for } 0 \leq x \leq 1.$$

More information about distribution functions of sequences can be found in the excellent treatise by Kuipers and Niederreiter, [5, pp. 53 and ff.].

It is known (see [5, pp. 137 and ff.]) that given any non-decreasing function, f, on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$, there exists a sequence in $[0,1]$ having f as its a.d.f. It can be even proved that any everywhere dense sequence in $[0,1]$ can be rearranged so as to yield a sequence having f as its a.d.f. (The proofs of these results are purely existential and not constructive.) Consequently, there exists a rearrangement of the sequence r_n of all rationals in $(0,1)$ with $?(x)$ as its a.d.f. We show one of these rearrangements to be the enumeration of the positive rationals obtained through their continued fraction development as we presented in [9]. In [10] we used a different enumeration of the rationals, based on Pierce expansions (see [16]), to present them as the a.d.f. of another interesting singular function.

In section 3 we prove the singularity of $?(x)$ by finding a new set on which the derivative is zero. This set is a different set from the set found by Salem, cited above.

Finally, in section 4, through the comparison of the 'normality' of numbers in $(0,1]$ as represented by continued fractions or by alternated dyadic fractions, we will specifically describe a set of measure one transformed by $?(x)$ into a set of measure zero and whose inverse image by $?(x)$ is also of measure zero. On the points of this set in which $?(x)$ has a derivative (a set of measure one) this derivative has to be, necessarily, 0, which proves again the singularity of Minkowski's function. A similar approach was used in [12] to the same end.

2 The enumeration of the rationals in (0,1)

We define a one–to–one correspondence, q between the set of positive integers, $\{1, 2, 3, \ldots\}$, and the set of all rational numbers in $(0,1)$ in the following way. If $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}$ with $0 \leq a_1 < a_2 < \cdots < a_k$,

$$q(n) = [0; a_1 + 1, a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1} + 1]. \tag{2}$$

This enumeration is a restriction to $(0,1)$ of a more general enumeration of all positive rationals (see [9]).

From now on, as we will only consider numbers in $(0,1)$ we will drop the 0 in the regular continued fraction representation of a number in $(0,1)$. Thus (2) will be written

$$q(n) = [a_1 + 1, a_2 - a_1, \ldots, a_k - a_{k-1} + 1].$$

A few terms of this enumeration are:

$q(1)$=[2]=1/2	$q(5)$=[1,3]=3/4	$q(9)$=[1,4]=4/5	$q(13)$=[1,2,2]=5/7
$q(2)$=[3]=1/3	$q(6)$=[2,2]=2/5	$q(10)$=[2,3]=3/7	$q(14)$=[2,1,2]=3/8
$q(3)$=[1,2]=2/3	$q(7)$=[1,1,2]=3/5	$q(11)$=[1,1,3]=4/7	$q(15)$=[1,1,1,2]=5/8
$q(4)$=[4]=1/4	$q(8)$=[5]=1/5	$q(12)$=[3,2]=2/7	$q(16)$=[6]=1/6

A careful observation of the enumeration provides the following facts about it, which are easily proved:

1. $q(2^n) = \dfrac{1}{n+2}$.

2. After $r/s, r/s < 1/2$ we have $(s-r)/s$, which amounts to say,

$$\frac{r}{s} = [a_1, a_2, \ldots] \text{ with } a_1 > 1 \quad \text{is followed by} \quad \frac{s-r}{s} = [1, a_1 - 1, a_2, \ldots].$$

3. The 2^{n-2} rationals, r/s, between places 2^{n-2} (included) and 2^{n-1} (excluded) are such that:

$$r/s = [a_1, a_2, \ldots, a_k], \ (a_k > 1) \quad \text{and} \quad \sum_{j=1}^{k} a_j = n. \tag{3}$$

   There are precisely 2^{n-1} possible partitions of a positive integer n in smaller positive integers if we consider different two partitions in which the order of the sumands is different (see problem 21 in Pólya and Szegö, [11]). If we ban those partitions in which the last sumand is 1, we get a a total of $2^{n-1} - 2^{n-2} = 2^{n-2}$ partitions coinciding with our 2^{n-2} rationals $q(2^{n-2}), q(2^{n-2}+1), \ldots, q(2^{n-1}-1)$.

It is immediate to see the following:

Lemma 2.1 *If we denote by $\sigma(x)$ the successor of x in the enumeration $\{q(n)\}$ then:*

1. $\sigma(1/2) = 1/3$.

2. If $x = [a_1, a_2, \ldots, a_k], \ (a_k > 1)$,

$$\sigma(x) = \begin{cases} [1, a_1 - 1, a_2, \ldots, a_k] & \text{if } a_1 > 1; \\ [h, a_h - 1, a_{h+1}, \ldots, a_k] & \text{if } a_1 = \ldots = a_{h-1} = 1, \ (h \leq k); \\ [h+3] & \text{if } x = [\underbrace{1, 1, \ldots, 1}_{h}, 2]. \end{cases}$$

This operator, σ, can be extended following the same formation rules to all real numbers in $(0,1)$ to define a partial order in all $(0,1)$.

2.1 An analytical expression for $\sigma(x)$

In the continued fraction expansion of the number $\Phi = (1/2)(\sqrt{5} - 1)$:

$$\Phi = [1, 1, 1, \ldots, 1, 1, \ldots]$$

let us consider its convergents:

$$R_0 = 0, \qquad R_i = [\underbrace{1, 1, \ldots, 1}_{i}] = [\underbrace{1, 1, \ldots, 1}_{i-2}, 2].$$

We have the following infinite chain of inequalities:

$$0 = R_0 < R_2 < R_4 < \cdots < \Phi < \cdots < R_5 < R_3 < R_1 = 1.$$

We can now consider the following family of half–open intervals, mutually disjoint, taken at left and right of Φ: on the left, $[R_{2k}, R_{2k+2})$ and on the right, $(R_{2k+1}, R_{2k-1}]$, such that, being mutually disjoint we have:

$$\bigcup_{k=0}^{\infty}[R_{2k}, R_{2k+2}) = [0, \Phi); \qquad \bigcup_{k=1}^{\infty}(R_{2k+1}, R_{2k-1}] = (\Phi, 1].$$

The function $\sigma(x)$ has the following piece–wise analytical expression:

$$\sigma(x) = \begin{cases} [k+1] = \dfrac{1}{k+1} & \text{if } x = R_k \\ \dfrac{F_{k+1}x - F_k}{(kF_{k+1} - F_k)x + F_{k-1} - kF_k} & \text{if } x \text{ is between } R_{k-1} \text{ and } R_{k+1} \end{cases}$$

where F_n is the Fibonacci sequence:

$$F_0 = 0;\ F_1 = 1;\ F_2 = 1;\ F_3 = 2;\ \ldots\ ; F_n = F_{n-1} + F_{n-2}.$$

The only point in $(0, 1)$ that lacks an image by σ is Φ. The graph of σ is shown in figure 1.

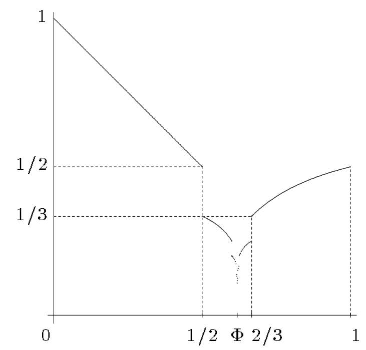

Figure 1: The graph of $\sigma(x)$

2.2 The distribution function of $\{q(n)\}$

We are going to prove that the a.d.f. of $\{q(n)\}$ is precisely $?(x)$. The proof we are going to give is a direct one, that is to say, we intend to see that given $x \in [0, 1]$:

$$\lim_{N\to\infty} \frac{\#\{q(i) \leq x; \quad i = 1, 2, \ldots, N\}}{N} = ?(x) \tag{4}$$

by calculating directly the limit in (4). We shall use the notation:

$$A(x; N) = \#\{q(i) \leq x; \quad i = 1, 2, \ldots, N\}.$$

The proof will be reached by different stages, first considering $x = 1/a = [a]$, and later by considering $x = [a_1, a_2, \ldots]$.

Lemma 2.2

$$A([a];2^M-1)=\begin{cases}0 & \textit{if} \quad a\geq M+2;\\ 2^{M-(a-1)} & \textit{otherwise.}\end{cases}$$

PROOF. It is seen at once that

$$[b_1,b_2,\ldots,b_k]\leq[a] \quad \text{iff} \quad b_1\geq a.$$

By the remark in (3) the rationals $q(1),q(2),\ldots,q(2^M-1)$ have continued fraction developments $[b_1,\ldots,b_k]$ verifying

$$\sum_{j=1}^{k} b_j\leq M+1.$$

As a consequence, if $a\geq M+2$, then there is no $[b_1,\ldots,b_k]$ such that $\sum b_j\leq M+1$ and $b_1\geq a$.

Now, if $a<M+2$, we are going to count $A([a];2^M-1)$ by blocks of 2^ℓ elements. the rationals between $q(2^\ell)$ (included) and $q(2^{\ell+1})$ (excluded) have expansions equal to $[b_1,\ldots,b_k]$ with $b_k>1$ and $\sum b_j=\ell+2$. Among these we must select those such that $b_1\geq a$, that is to say those of the forms:

$[a,b_2,\ldots,b_k]$, $(b_k>1)$, $\sum_{j=2}^k b_j=\ell+2-a$: by (3), a total of $2^{\ell-a}$
$[a+1,b_2,\ldots,b_k]$, $(b_k>1)$, $\sum_{j=2}^k b_j=\ell+1-a$: by (3), a total of $2^{\ell-a-1}$
$[a+2,b_2,\ldots,b_k]$, $(b_k>1)$, $\sum_{j=2}^k b_j=\ell-a$: by (3), a total of $2^{\ell-a-2}$
...
$[\ell,2]$ which amounts only to 1,
$[\ell+1,1]$ which is not admissible,
$[\ell+2]$ which amounts only to 1,

All in all:

$$1+(1+2+2^2+\cdots+2^{\ell+2-a})=2^{\ell+1-a}.$$

The block of rationals between $q(2^{a-2})$ and $q(2^{a-1}-1)$ for which $\sum b_j=a$ contribute with 1 element to the total count. For the rest of blocks we have a total of:

$$\sum_{\ell=a-1}^{M-1}2^{\ell+1-a}=2^{M-a+1}-1.$$

Finally, we have a total of:

$$A([a];2^M-1)=1+(2^{M-a+1}-1)=2^{M-(a-1)}. \qquad \square$$

Now, a careful observation of the enumeration leads us to a new result which is very significative. Within a block $q(2^\ell),\ldots,q(2^{\ell+1}-1)$ in which

$$q(i)=[b_1,\ldots,b_k],\ (b_k>1),\ \sum b_j=\ell+2,$$

the 2^ℓ $q(i)$'s distribute themselves following again the same pattern as the $2^{\ell+1}$ from $q(1)$ to $q(2^{\ell+1}-1)$. To see that, you only have to drop the last b_k and add

1 to b_{k-1} :

$$\begin{array}{lll}
\text{First one:} & \text{sum } 2 \left\{ \; q(2^\ell+1)=[1,\ell+1] \right. & \longrightarrow \; [2] \\
\text{From } 2^1 \text{ to } 2^2-1 : \text{sum } 3 & \left\{ \begin{array}{ll} q(2^\ell+2)=[2,\ell] & \longrightarrow \; [3] \\ q(2^\ell+3)=[1,1,\ell] & \longrightarrow \; [1,2] \end{array} \right. & \\
\text{From } 2^2 \text{ to } 2^3-1 : \text{sum } 4 & \left\{ \begin{array}{ll} q(2^\ell+4)=[3,\ell-1] & \longrightarrow \; [4] \\ q(2^\ell+5)=[1,2,\ell-1] & \longrightarrow \; [1,3] \\ q(2^\ell+6)=[2,1,\ell-1] & \longrightarrow \; [2,2] \\ q(2^\ell+7)=[1,1,1,\ell-1] & \longrightarrow \; [1,1,2] \end{array} \right. & \\
\text{From } 2^3 \text{ to } 2^4-1 : \text{sum } 5 & \left\{ \begin{array}{ll} q(2^\ell+8)=[4,\ell-2] & \longrightarrow \; [5] \\ \ldots\ldots & \ldots \\ q(2^\ell+15)=[1,1,1,1,\ell-2] & \longrightarrow \; [1,1,1,2] \end{array} \right. & \\
\ldots\ldots & &
\end{array} \tag{5}$$

This pattern is repeated at deeper levels as we get from right to left in the continued fraction expansion of the $q(i)$'s: each block of 2^ℓ $q(i)$'s break in smaller blocks of 2^j elements sharing their last coefficient.

With this last observation, it is easily (but tediously) proved the next lemma:

Lemma 2.3 *If $N = 2^{B_0} + 2^{B_1} + \cdots + 2^{B_t}$ with $0 \leq B_0 < B_1 < \ldots < B_t$ is the dyadic expression of N, then*

$$A([a];N) = \begin{cases} 0 & \textit{if} \quad a > B_k + 2; \\ 1 + \left\lfloor \sum_{j=0}^{t} 2^{B_j-(a-1)} \right\rfloor & \textit{otherwise.} \end{cases} \tag{6}$$

(The symbols $\lfloor \; \rfloor$ denote the integer part.)

In point of fact, some of the sumands within the sum in (6) are of the form 2^{-n} if $a < B_j + 1$ but as their contribution to the total sum will never equal or exceed 1, it is easier to use the above formula than making exceptions.

With the help of this last lemma it is easy to see the following:

Theorem 2.4
$$\lim_{N\to\infty} \frac{A([a];N)}{N} = ?(1/a).$$

Proof. By lemma 2.3:

$$\begin{aligned}
&\lim_{N\to\infty} \frac{A([a];N)}{N} = \\
&\quad = \lim_{N\to\infty} \frac{1 + \sum_{B_j > a+1} 2^{B_j-(a-1)}}{\sum_{j=0} 2^{B_j}} = \\
&\quad = \lim_{N\to\infty} \frac{1 + \sum_{j=0}^{t} 2^{B_j-(a-1)} - \sum_{B_j \leq a+1} 2^{B_j-(a-1)}}{\sum_{j=0} 2^{B_j}} = \\
&\quad = \lim_{N\to\infty} \left(\frac{1 + 2^{-(a-1)}N}{N} - 2^{-(a-1)} \cdot \frac{\sum_{B_j \leq a+1} 2^{B_j}}{N} \right) = \frac{1}{2^{a-1}}. \quad \square
\end{aligned}$$

Now we generalize the above result to an $x = [a_1, a_2, \ldots, a_h]$. To start with, let us consider $x = [a_1, a_2]$. Now, $[b_1, b_2, \ldots] < [a_1, a_2]$ either if $b_1 > a_1$ or $b_1 = a_1$ and $b_2 < a_2$. With this observation, it is easy to see that

$$A([a_1,a_2];2^M-1) = A([a_1];2^M-1) - A([a_1+a_2];2^M-1) + \varepsilon, \tag{7}$$

where ε is 0 or 1,

This is because from all the $q(i) = [b_1, \ldots, b_k]$ which verify $b_1 \geq a_1$ we must exclude those which verify $b_1 = a_1$ and $b_2 \geq a_2$ with the only exception of the very $[a_1, a_2]$. Now, those rationals between $q(1)$ and $q(2^M - 1)$, which verify $b_1 = a_1$ and $b_2 \geq a_2$ can be counted by blocks in the same way we did above in the proof of lemma 2.2. We are going to count the rationals between $q(2^\ell)$ (included) and $q(2^{\ell+1})$ (excluded) with expansions equal to $[a_1, b_2, b_3, \ldots, b_k]$ with $b_k > 1$ and $b_2 \geq a_2$. For these, $a_1 + \sum b_j = \ell + 2$. Among these we must select those of the forms:

$$\begin{aligned}
&[a_1, a_2, b_3, \ldots, b_k], \; (b_k > 1), \; \textstyle\sum_{j=3}^k b_j = \ell + 2 - (a_1 + a_2) : \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by (3), a total of } 2^{\ell-(a_1+a_2)} \\
&[a_1, a_2 + 1, b_3, \ldots, b_k], \; (b_k > 1), \; \textstyle\sum_{j=3}^k b_j = \ell + 1 - (a_1 + a_2) : \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by (3), a total of } 2^{\ell-(a_1+a_2)-1}
\end{aligned}$$

etc. as before.

All in all, it is the same count we would do in order to find out those rationals between $q(2^\ell)$ and $q(2^{\ell+1} - 1)$ whose first term, $b_1 \geq a_1 + a_2$, that is to say, $A([a_1 + a_2], 2^M - 1)$. The value of ε, 0 or 1, can be precised through a more detailed analysis of the problem but it is irrelevant for our purpose. $\square$

Equation (7) can be easily generalized to

$$A([a_1, a_2]; N) = A([a_1]; N) - A([a_1 + a_2]; N) + \varepsilon, \quad (\varepsilon = 0, 1),$$

again through observing the pattern described in (5).

This analysis can be carried further and provides us with the result:

Lemma 2.5 *If $N = 2^{B_0} + 2^{B_1} + \cdots + 2^{B_t}$ with $0 \leq B_0 < B_1 < \ldots < B_t$ is the dyadic expression of N, we have:*

$$\begin{aligned}
A([a_1, a_2, \ldots, a_h]; N) &= \\
&= A([a_1]; N) - A([a_1 + a_2]; N) + A([a_1 + a_2 + a_3]; N) - \cdots = \\
&= \sum_{j=1}^{h} (-1)^{j-1} A([a_1 + a_2 + \cdots + a_j]; N) + \varepsilon, \quad (\varepsilon = -1, 0, 1). \qquad (8)
\end{aligned}$$

Finally, by this former lemma, it is seen at once:

Theorem 2.6

$$\begin{aligned}
\lim_{N \to \infty} \frac{A([a_1, a_2, \ldots, a_h]; N)}{N} &= \\
&= \frac{1}{2^{a_1 - 1}} - \frac{1}{2^{a_1 + a_2 - 1}} + \cdots + \frac{(-1)^{h+1}}{2^{a_1 + a_2 + \cdots + a_h - 1}} = \\
&= ?([a_1, \ldots, a_h]).
\end{aligned}$$

And, by continuity, the final theorem we were seeking,

Theorem 2.7

$$\lim_{N \to \infty} \frac{A([a_1, a_2, \ldots]; N)}{N} = \frac{1}{2^{a_1 - 1}} - \frac{1}{2^{a_1 + a_2 - 1}} + \cdots = ?([a_1, a_2, \ldots]).$$

3 The singularity of $?(x)$

We are going to exhibit a set of measure one on which $?(x)$ has a zero derivative. This set is going to be quite different from the one presented by Salem in [15]. Salem starts with the set of all numbers in $[0,1]$ whose regular continued fraction expansion had unbounded partial quotients and shows that at the points of this set, $?'(x)$ is either 0 or ∞. Limiting himself to the points in which $?'(x) = 0$ he gets the set of measure one he seeks.

Our starting set will also be described using some specific metrical properties of the regular continued fraction expansion of a real number, but the main difference with Salem's set will be that at the points of our set, $?'(x) = 0$ whenever it exists in a broad sense ($?'(x) \leq \infty$).

3.1 The continued fraction system of representation

In the regular continued fractions system of representation, limited to numbers in $(0,1]$, the residue function can be defined as:

$$R(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

In a certain sense, Kuzmin proved in [6] that for almost all x in $(0,1]$, the a.d.f. of the sequence $\{x, R(x), R^2(x), R^3(x), \ldots\}$ is $\log_2(1+x)$. This result is the consequence of an unproved conjecture of Gauss and, since Kuzmin's proof, it has been known as the Gauss–Kuzmin theorem (see [13, Chap. V] for more details).

It can be seen that the residue function $R(x)$ preserves Gauss's measure, whose density is precisely:

$$\mu(x) = \log_2(1+x).$$

A number $x \in (0,1]$ whose orbit $\{x, R(x), R^2(x), R^3(x), \ldots\}$ has $\log_2(1+x)$ as its a.d.f. will be called a *Gauss–Kuzmin number.*

It is well–known that the set of $x \in (0,1]$ for which the mean value of their partial quotients, $(a_1 + \cdots + a_n)/n$ tends to ∞ is a set of measure one (see [3, 13] for more details). In the next theorem we are a bit more precise, and we prove that the set of Gauss–Kuzmin numbers is a susbset of this one.

Theorem 3.1 *If $x = [a_1, a_2, a_3, \ldots]$ is a Gauss–Kuzmin number then*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \infty.$$

PROOF. If $x = [a_1, a_2, a_3, \ldots]$ is a Gauss–Kuzmin number it is seen at once that the frequency of repetitions of the number i among the partial quotients, or, for short, the *density* of i, is

$$\lim_{n\to\infty} \frac{\#\{a_j = i;\ j = 1, 2, \ldots, n\}}{n} = \log_2\left(\frac{(i+1)^2}{i(i+2)}\right) = p(i).$$

That means that given a positive integer i and given $\varepsilon > 0$, there exists a n_i such that for all $n \geq n_i$ we have:

$$\left| \frac{\#\{a_j = i;\ j = 1, 2, \ldots, n\}}{n} - p(i) \right| < \varepsilon. \tag{9}$$

Now, let k be given and let $n_0 = \max(n_1, n_2, \ldots, n_{k-1})$. We define $\bar{x} = [b_1, b_2, \ldots, b_n, \ldots]$ as follows:

$$\begin{cases} b_i = a_i & \text{if } a_i \leq k; \\ b_i = k & \text{if } a_i > k. \end{cases}$$

Obviously, for all n we have:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \frac{b_1 + b_2 + \cdots + b_n}{n}.$$

The sequence $\{b_n\}$ takes values only in the set $\{1, 2, \ldots, k\}$ and if x was a Gauss–Kuzmin number then the density of number $i, 1 \leq i \leq k-1$ is $p(i)$ whereas the density of number k is

$$\bar{p}(k) = \sum_{i=k}^{\infty} \log_2 \left(\frac{(i+1)^2}{i(i+2)} \right) = \log_2 \frac{k+1}{k}.$$

Thus, given $\varepsilon > 0$, there exists a n_k such that for all $n \geq n_k$ we have:

$$\left| \frac{\#\{b_j = k; \, j = 1, 2, \ldots, n\}}{n} - \bar{p}(k) \right| < \varepsilon. \tag{10}$$

If $n \geq \bar{m} = \max(n_0, n_k)$, both (9) and (10) hold, and we have:

$$\begin{aligned}
&\frac{b_1 + b_2 + \cdots + b_n}{n} \geq \\
&\geq \sum_{i=1}^{k-1} i \, (p(i) - \varepsilon) + k \, (\bar{p}(k) - \varepsilon) = \\
&= \sum_{i=1}^{k-1} i \log_2 \left(\frac{(i+1)^2}{i(i+2)} \right) + k \cdot \log_2 \frac{k+1}{k} - \sum_{i=1}^{k} i\varepsilon = \\
&= \log_2 \prod_{i=1}^{k-1} \left[\frac{i+1}{i} \cdot \frac{i+1}{i+2} \right]^i \left(\frac{k+1}{k} \right)^k - \varepsilon \frac{k(k+1)}{2} = \\
&= \log_2 \left[\left(\frac{2}{1} \cdot \frac{2}{3} \right) \left(\frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \right) \left(\frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \right) \cdots \right] - \varepsilon \frac{k(k+1)}{2} = \\
&= \log_2(k+1) - \varepsilon \frac{k(k+1)}{2}.
\end{aligned}$$

Therefore, if given k a positive integer, we take $\varepsilon = \frac{2}{k(k+1)}$, there exists $\bar{m}(k)$ such that for all $n \geq \bar{m}(k)$ we have:

$$\frac{a_1 + \cdots + a_n}{n} \geq \frac{b_1 + \cdots + b_n}{n} \geq \log_2(k+1) - 1, \tag{11}$$

which implies

$$\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = \infty. \qquad \square$$

Let D denote the set of $x \in [0,1]$ for which $?'(x) \leq \infty$. And let G denote the set of $x \in (0,1)$ of Gauss–Kuzmin numbers. Besides, we will consider the set K of $x \in (0,1)$ whose continued fraction expansion verify the Khintchine–Lévy constant (see [13, Chap. V]), that is, such that if $x = [a_1, a_2, \ldots, a_n, \ldots]$ and p_n/q_n is the sequence of its convergents then

$$\lim_{n\to\infty} \sqrt[n]{q_n} = e^{\frac{\pi^2}{12\log 2}}.$$

or, what amounts to the same,

$$\lim_{n\to\infty} \frac{\log q_n}{n} = \frac{\pi^2}{12\log 2}.$$

We will call these numbers *Khintchine–Lévy numbers.* These three subsets of $[0,1]$: D, G and K are of measure one.

Theorem 3.2 *If $x \in D \cap G \cap K$ then $?'(x) = 0$.*

Proof. Let $x = [a_1, a_2, \ldots, a_n, \ldots]$ and $R_n = p_n/q_n$ be the sequence of its convergents. We know that, if n is even, $R_n < x < R_{n-1}$; then, as

$$?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \frac{1}{2^{a_1+a_2+a_3-1}} - \cdots,$$

if $x \in D$,

$$\begin{aligned} ?'(x) &= \lim_{n\to\infty} \frac{?(R_{n-1}) - ?(R_n)}{R_{n-1} - R_n} = \\ &= \lim_{n\to\infty} \frac{\frac{1}{2^{a_1+a_2+\cdots+a_n-1}}}{\frac{1}{q_n q_{n-1}}} = \lim_{n\to\infty} \frac{q_n q_{n-1}}{2^{a_1+a_2+\cdots+a_n-1}}. \end{aligned} \tag{12}$$

We must see that if, besides, $x \in G \cap K$ then this last limit is 0.

Taking logarithms in the sequence of the limit (12) we seek,

$$\begin{aligned} &\log q_n + \log q_{n-1} - (a_1 + \cdots + a_n - 1)\log 2 = \\ &= n \cdot \left[\frac{\log q_n + \log q_{n-1}}{n} - \frac{(a_1 + \cdots + a_n - 1)}{n} \cdot \log 2\right] \to -\infty \end{aligned} \tag{13}$$

as

$$\frac{\log q_n + \log q_{n-1}}{n} \to 2\frac{\pi^2}{12\log 2},$$

and, by theorem 3.1,

$$\frac{a_1 + a_2 + \cdots + a_n - 1}{n} \to \infty.$$

The limit in (13) proves that $?'(x) = 0$. □

A closer look at the proof we have just seen, shows that the condition of $x \in K$ can be lightened. It is enough for our purposes that the expression within brackets in (13) tends to $-\infty$ so that, in the end, the whole limit in (13) tends to $-\infty$. This requirement can be fulfilled just by the condition of x being a Gauss–Kuzmin number, as our next theorem proves:

Theorem 3.3 *If $x \in D \cap G$ then $?'(x) = 0$.*

Proof. We get, as before, that if $x \in G \cap D$,

$$?'(x) = \lim_{n\to\infty} \frac{q_n q_{n-1}}{2^{a_1+a_2+\cdots+a_n-1}},$$

and, taking logarithms in this last limit:

$$\begin{aligned}\log ?'(x) &= \\ &= \lim_{n\to\infty} (\log q_n + \log q_{n-1} - (a_1 + \cdots a_n - 1)\cdot \log 2) = \\ &= \lim_{n\to\infty} n \cdot \left[\frac{\log q_n + \log q_{n-1}}{n} - \log 2 \cdot \frac{a_1 + \cdots a_n - 1}{n}\right]. \end{aligned} \tag{14}$$

Now, as p_n/q_n are the convergents of the continued fraction $[a_1, a_2, \ldots, a_n, \ldots]$, the q_n satisfy the recurrence,

$$q_n = a_n q_{n-1} + q_{n-2}; \qquad q_0 = 1,\ q_1 = a_1,$$

and, trivially,

$$q_n < (a_n + 1)(a_{n-1} + 1)\cdots(a_1 + 1).$$

Going back to the expression in (14),

$$\begin{aligned}&\frac{\log q_n + \log q_{n-1}}{n} - \log 2 \cdot \frac{a_1 + \cdots + a_n - 1}{n} < \\ &\quad < 2 \cdot \frac{\log q_n}{n} - \log 2 \cdot \frac{a_1 + \cdots + a_n - 1}{n} < \\ &\quad < 2 \cdot \frac{\sum_{j=1}^n \log(a_j + 1)}{n} - \log 2 \cdot \frac{a_1 + \cdots + a_n - 1}{n}.\end{aligned}$$

As we did before in the proof of theorem 3.1, given a positive integer k let us replace a_n by b_n where

$$\begin{cases} b_i = a_i & \text{if } a_i \le k; \\ b_i = k & \text{if } a_i > k; \end{cases}$$

We will need two lemmas to go on:

Lemma 3.4 *The function*

$$f(x) = \log \frac{(x+1)^2}{2^x}$$

is strictly decreasing for $x \ge 2$.

Lemma 3.5 *The series*

$$\sum_{r=1}^{\infty} \log(r+1) \cdot \log \frac{(r+1)^2}{r(r+2)}$$

converges to a positive value, η.

Both are proved trivially.

Now, if k is large enough for lemma 3.4 to be valid,

$$\begin{aligned}&2 \cdot \frac{\sum_{j=1}^n \log(a_j + 1)}{n} - \log 2 \cdot \frac{a_1 + \cdots + a_n - 1}{n} < \\ &\quad < 2 \cdot \frac{\sum_{j=1}^n \log(b_j + 1)}{n} - \log 2 \cdot \frac{b_1 + \cdots + b_n - 1}{n}\end{aligned} \tag{15}$$

Besides, using the results we obtained in the proof of theorem 3.1:

$$\frac{b_1 + b_2 + \cdots + b_n - 1}{n} \geq \log(k+1) - 1 \tag{16}$$

and, given $\varepsilon = \frac{2}{k(k+1)}$, for n large enough, both (9) and (10) were valid.

Consequently, the inequality obtained in (15) can be continued. For n large enough:

$$\begin{aligned}
2 \cdot \frac{\sum_{j=1}^{n} \log(b_j + 1)}{n} - \log 2 \cdot \frac{b_1 + \cdots + b_n - 1}{n} &\leq \\
\leq 2 \sum_{i=1}^{k-1} \log(i+1) \left(\log_2 \frac{(i+1)^2}{i(i+2)} + \varepsilon \right) &+ \\
+2 \log(k+1) \cdot (\log_2 \frac{k+1}{k} + \varepsilon) - \log 2 \cdot (\log(k+1) - 1) &= \\
= 2 \sum_{i=1}^{k-1} \log(i+1) \left(\log_2 \frac{(i+1)^2}{i(i+2)} \right) + 2\varepsilon \sum_{i=1}^{k} \log(i+1) &+ \\
+2 \log(k+1) \cdot \log_2 \frac{k+1}{k} - \log 2 \cdot (\log(k+1) - 1)
\end{aligned}$$

Now, remembering that $\varepsilon = \frac{2}{k(k+1)}$,

$$\begin{aligned}
2\varepsilon \sum_{i=1}^{k} \log(i+1) &\leq \\
&\leq \frac{4}{k(k+1)} \cdot \int_1^{k+1} \log(x+1) dx \leq \\
&\leq \frac{4}{k(k+1)} \cdot (k+2) \cdot \log(k+2),
\end{aligned}$$

which tends to 0 as $k \to \infty$.

On the other hand, by lemma 3.5

$$2 \sum_{i=1}^{k-1} \log(i+1) \left(\log_2 \frac{(i+1)^2}{i(i+2)} \right) \leq 2 \sum_{i=1}^{\infty} \log(i+1) \left(\log_2 \frac{(i+1)^2}{i(i+2)} \right) = 2\eta.$$

All in all, we have:

$$\begin{aligned}
2 \sum_{i=1}^{k-1} \log(i+1) \left(\log_2 \frac{(i+1)^2}{i(i+2)} \right) + 2\varepsilon \sum_{i=1}^{k} \log(i+1) &+ \\
+2 \log(k+1) \cdot \log_2 \frac{k+1}{k} - \log 2 \cdot (\log(k+1) - 1) &\leq \\
\leq 2\eta + \frac{4}{k(k+1)} \cdot (k+2) \log(k+2) + 2 \log(k+1) \cdot \log_2 \frac{k+1}{k} &- \\
- \log 2 \cdot (\log(k+1) - 1) &= \\
= 2\eta + \frac{4}{k(k+1)} \cdot (k+2) \log(k+2) &+ \\
+ \log(k+1) \cdot \left(2 \log_2 \frac{k+1}{k} - \log 2 - \frac{1}{\log(k+1)} \right).
\end{aligned}$$

This last expression, clearly tends to $-\infty$ when $k \to \infty$.

Summing up,

$$\left[\frac{\log q_n + \log q_{n-1}}{n} - \log 2 \cdot \frac{a_1 + \cdots + a_n}{n}\right] \cdot n \to -\infty. \qquad \square$$

4 A 'vanishing' set under $?(x)$

In this section we are going to prove the singularity of $?(x)$ by finding what we call a *vanishing* set, that is, a set of measure one whose image under $?(x)$ is of measure zero and whose inverse image is also of measure zero. On the points of this set for which $?'(x)$ exists, we must have $?'(x) = 0$.

4.1 The alternated dyadic system

The expansions in Salem's expression (1) of $?(x)$ constitute an instance of a peculiar system of representation of real numbers, *the alternated dyadic system.* As we are going to use it in this section and it is not very well-known, it is worth our while to examine its most important features.

Theorem 4.1 *Any real number in* $[0, 1]$ *can be represented in an unique way (except for the duplicity of terminating expansions) as:*

$$x = \frac{1}{2^{d_1}} - \frac{1}{2^{d_2}} + \cdots + \frac{(-1)^{n+1}}{2^{d_n}} + \cdots \tag{17}$$

where $\{d_i\}$ *are a strictly increasing sequence of non–negative integers,* $0 \leq d_1 < d_2 < \cdots < d_n < \cdots$.

We sketch the proof of this theorem. The sequence $\{1/2^n\}$ induces a partition of $(0, 1]$:

$$(0, 1] = \bigcup_{n=0}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]. \tag{18}$$

Given x, there exists a positive integer, n such that:

$$\frac{1}{2^{n+1}} < x \leq \frac{1}{2^n},$$

from where we find

$$n = \left\lfloor \log_2 \frac{1}{x} \right\rfloor. \tag{19}$$

Thus, x can be written as:

$$x = \frac{1}{2^n} - \lambda\left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right) = \frac{1}{2^n} - \lambda\frac{1}{2^{n+1}} = \frac{1}{2^n}\left(1 - \frac{\lambda}{2}\right), \quad \text{with } \lambda \in [0, 1). \tag{20}$$

From this last equality we get $2^{n+1}x = 2 - \lambda$ and thus $\lambda = 2(1 - 2^n x)$.

Now, we define the residue function as

$$F(x) = 2(1 - 2^n x), \qquad \text{where } n = \left\lfloor \log_2 \frac{1}{x} \right\rfloor,$$

and with its help we obtain the recurrence that provides the different terms of the expansion:

$$\begin{cases} \omega_1 = x \\ d_1 = \left\lfloor \log_2 \frac{1}{x} \right\rfloor \end{cases} ; \quad \begin{cases} \omega_n = F(\omega_{n-1}) \\ d_n = 1 + \left\lfloor \log_2 \frac{1}{\omega_n} \right\rfloor + d_{n-1} \end{cases} \quad \text{for } n > 1. \tag{21}$$

Lemma 4.2 *The residue function $F(x)$ preserves the Lebesgue measure, λ.*

PROOF. Let $y \in [0,1]$ and let us consider the set $A(y) = \{x : F(x) \leq y\}$. For each interval in the dyadic partition (18) we will have $y = 2(1 - 2^n x)$; that is to say, $x = \frac{1}{2^n} - \frac{y}{2^{n+1}}$. Consequently, the numbers x such that $F(x) \leq y$ define for each n a subinterval $\left(\frac{1}{2^n} - \frac{y}{2^{n+1}}, \frac{1}{2^n}\right]$. Therefore (see figure 2):

$$\lambda(A(y)) = \sum_{n=0}^{\infty} \frac{y}{2^{n+1}} = y. \qquad \square$$

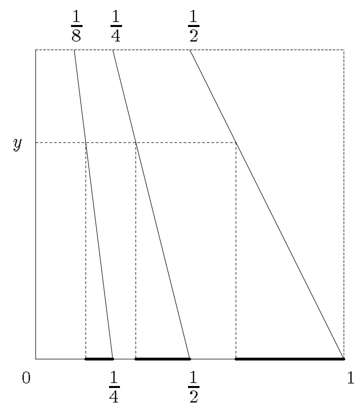

Figure 2: The residue function

4.2 Normal numbers to the alternated dyadic system

Definition 1 *We will say that a number x is* normal *to the alternated dyadic system given by (17) when its orbit under F, $\{x, F(x), F^2(x), F^3(x), \ldots\}$ is uniformly distributed in $(0,1]$.*

The definition is analogous to the one given by Wall in [17], (see also [8, Chap. 8] or [5, Chap. 1, Sect. 8]) for the usual integer–based systems of representation, which is equivalent to the classic one by Borel.

Now, $F(x)$ preserves Lebesgue's measure, as we proved in lemma 4.2 and it can be proved by means of Knopp's theorem (see [4]) to be an ergodic function. Consequently, the orbits $\{x, F(x), F^2(x), F^3(x), \ldots\}$ are uniformly distributed for almost all x in $(0,1]$ (for a discussion of these topics from the ergodic point of view, see [1]), and thus the set of *normal* numbers to the alternated dyadic system is a set of measure one.

Theorem 4.3 *If x is a normal number to the alternated dyadic system, we have:*

$$\lim_{n\to\infty} \frac{d_n(x)}{n} = 2.$$

PROOF. By the recurrence (21) we have:

$$d_n(x) = d_1(x) + \sum_{j=2}^{n}(d_j - d_{j-1}) = n - 1 + \sum_{j=1}^{n} \left\lfloor \log_2 \frac{1}{\omega_i} \right\rfloor .$$

If x is normal, the sequence $\{\omega_i\}$ is uniformly distributed and thus the relative frequency of visits of ω_i in the interval $\left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]$ tends to be equal to $\frac{1}{2^{k+1}}$. For these ω_i, $\left\lfloor \log_2 \frac{1}{\omega_i} \right\rfloor = k$. This means that, for n large

$$\sum_{j=1}^{n} \left\lfloor \log_2 \frac{1}{\omega_i} \right\rfloor \approx \sum_{k=0}^{\infty} k \cdot \frac{n}{2^{k+1}} = n \cdot \sum_{k=0}^{\infty} \frac{k}{2^{k+1}} = n.$$

Therefore,

$$\lim_{n\to\infty} \frac{d_n(x)}{n} = \lim_{n\to\infty} \left(\frac{n-1}{n} + \frac{1}{n} \cdot \sum_{j=1}^{n} \left\lfloor \log_2 \frac{1}{\omega_j} \right\rfloor \right) = 2. \qquad \square$$

It is easily seen that the converse of theorem 4.3 is not true.

4.3 A vanishing set

Let us now consider the two sets, G of Gauss–Kuzmin numbers, and N of normal numbers to the alternated dyadic system. Both have measure one, so their intersection, $G \cap N$, has also measure one.

Theorem 4.4 $\lambda(?(G \cap N)) = 0$ *and* $\lambda(?^{-1}(G \cap N)) = 0$

PROOF. Minkowski's $?(x)$ function maps $[0,1]$ one to one onto itself and $?([a_1, a_2, \ldots])$ is written with the digits $d_n = a_1 + \cdots + a_n$ in the alternated dyadic system. Now, if $x \in G$, by theorem 3.1

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \lim_{n\to\infty} \frac{d_n}{n} = \infty.$$

Therefore, according to theorem 4.3, $?(x)$ is not a normal number to the alternated dyadic system, which implies:

(A) the set $?(G)$ is a set of measure zero.

Besides, if $?(x) \in N$, then

$$\lim_{n\to\infty} \frac{d_n}{n} = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 2,$$

and thus, by theorem 3.1, x cannot be a Gauss–Kuzmin number and we have:

(B) the set $?^{-1}(N)$ is a set of measure zero.

(A) and (B) prove that $G \cap N$ is a set of measure one such that both $?(G \cap N)$ and $?^{-1}(G \cap N)$ are sets of measure zero. $\square$

5 Conclusions

Salem's presentation of $?(x)$ is shown to be the asymptotic distribution function of an enumeration of the rationals in $(0, 1]$ based on their expansion as regular continued fractions. Besides, Salem's expression links two systems for real number representation: regular continued fractions and the alternated dyadic system. This link permits to establish the singularity of Minkowski's function by studying the transformation of sets defined in (0,1) through metrical properties of the two systems of representation.

References

[1] P. Billingsley. *Ergodic Theory and Information.* John Wiley & Sons, N.Y., 1965.

[2] A. Denjoy. "Sur une fonction réelle de Minkowski". *C.R. Acad. Sci. Paris.* **194** (1932), pp. 44–46. Also in *J. Math. Pures Appl.* **17** (1938), pp. 105–151.

[3] A. Ya. Khintchine. *Continued Fractions.* Translated by Peter Wynn. Noordhoff Ltd., Groningen, The Netherlands, 1963.

[4] K. Knopp. "Mengentheoretische Behandlung einiger Probleme der diophantischen Approximationen und der transfiniten Wahrscheinlichkeiten". *Math. Annalen*, **95** (1926), pp. 409–426.

[5] L. Kuipers & H. Niederreiter. *Uniform distribution of sequences.* John Wiley & Sons, N.Y., 1973.

[6] R.O. Kuzmin. "Sur un problème de Gauss". *Atti Congr. Intern. Bologne*, **6** (1928), pp. 83–89.

[7] H. Minkowski. *Gesammelte Abhandlungen* vol. 2 (1991), pp. 50–51.

[8] I. Niven. *Irrational Numbers*, Carus Math. Monographs, 11. Wiley, NY, 1956.

[9] J. Paradís, Ll. Bibiloni, P. Viader. "On Actually Computable Bijections between $\mathbb{N}$ and Q^+". *Order*, **13** (1996), pp. 369–377.

[10] J. Paradís, P. Viader, Ll. Bibiloni. "A total order in $[0, 1]$ defined through a 'next' operator". *Order*, Submitted October 1996.

[11] G. Pólya & G. Szegö. *Problems and Theorems in Analysis*, Vol I, Springer–Verlag, 1979.

[12] G. Ramharter. "On Minkowski's Singular Function". *American Math. Soc.*, **99**, Number 3, March, (1987), Shorter Notes Section.

[13] A. Rockett & P. Szüsz. *Continued Fractions.* World Scientific, Singapore, 1992.

[14] F. Ryde. "On the Relation between Two Minkowski Functions". *Journal of Number Theory*, **17** (1983), pp. 47–51.

[15] R. Salem. "On some singular monotonic functions which are strictly increasing". *Trans. Amer. Math. Soc.* **53** (1943), pp. 427–439.

[16] O.J. Shallit. "Metric Theory of Pierce Expansions". *Fibonacci Quarterly* **24** (1986), pp. 22–40.

[17] D.D. Wall. *Normal Numbers*. PhD Tesis. Univ. of California, Berkeley. 1949.