# On Bounds for Network Revenue Management

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**Abstract**

Network Revenue Management can be formulated as a stochastic dynamic programming problem ("optimal" solution) whose exact solution is computationally intractable. Consequently, a number of heuristics have been proposed in the literature, the most popular of which are the deterministic linear programming (DLP) model, and the randomized linear programming (RLP) model, both of which give upper bounds on the optimal solution value. These bounds are used to provide control values that can be used in practice to make accept/deny decisions for booking requests. Recently Adelman [1] and Topaloglu [14] have proposed alternate upper bounds and showed that their bounds are tighter than the DLP bound. Tight bounds are of great interest as it appears from empirical studies and practical experience that models that give tighter bounds also lead to better controls (better in the sense that they lead to more revenue). In this paper we prove relationships between all these bounds. Specifically, we show that Adelman's bound is weaker than the bound given by the RLP model (we call the perfect hind sight or $PH$ bound), and that the RLP bound is weaker than Topaloglu's bound. To summarize our paper ranks the bounds as follows: DLP $\geq$ Adelman's affine relaxation ($AR$) bound $\geq$ $PHIP$ and $PHLP$ $\geq$ Topaloglu's Lagrangian ($LR$) bound $\geq$ Optimal Solution.

*Key words.* revenue management, bid-prices, classification.

## 1 Introduction

Revenue management is the control of the sale of a limited quantity of a resource (hotel rooms for a night, airline seats, advertising slots etc.) to a heterogenous population with different valuations for the resource. The resource is perishable, and for simplicity sake, we assume that it perishes at a fixed point of time in the future. Customers are independent of each other and arrive randomly during a sale period, and demand one unit of resource each. Sale is online, so the firm has to decide if it wishes to sell or not at the requested price,

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the tradeoff being selling too much at too low a price early and running out of capacity, or, rejecting too many low valuation customers and ending up with excess unsold inventory. That is a brief description of revenue management. The reader should consult the books Talluri and van Ryzin [12] or Phillips [7] or the survey articles of McGill and van Ryzin [6], Elmaghraby and Keskinocak [4], and Bitran and Caldentey [3] for a thorough background on the existing theory and a survey of applications of revenue management.

In industries such as hotels and airlines inventory is sold as a bundle of different resources (multi-night stays, multi-leg itineraries) and the decision to accept or reject a particular product at a certain price depends on the future demands and revenues for all the resources used by the product (and also indirectly, all the resources in the network). Network revenue management (network RM) is control based on the demands for the entire network. Again, Talluri and van Ryzin [12] contains all the necessary background on RM and network RM (Chapter 3 of [12]).

The network revenue management problem, under certain assumptions on the demand process, can be formulated as a stochastic dynamic programming problem. The controls of this dynamic program is a subset of products to open up for sale at any given point in time. The state is time remaining and the remnant capacities on the resources. This dynamic program is practically impossible to solve for even small networks. So a number of heuristics have been devised that generated controls to make the accept/reject decisions, the oldest and most well-known and widely used of which are the the Deterministic Linear Programming (DLP), proposed by Simpson [8] and analyzed in Williamson [17], and the Randomized Linear Programming (RLP) method in [10], and the Displacement Adjusted Virtual Nesting (DAVN) heuristic of Belobaba [2]. The mathematical programs on which DLP and RLP are based can be shown to form upper bounds on the optimal solutions. We call these the DLP bound and the Perfect-hind sight ($PH$) bound respectively. The PH bound has two versions depending on whether one solves the integer programs ($PHIP$) or the linear programs ($PHLP$).

Recently, Adelman [1] and Topaloglu [14] have proposed new heuristics for network RM, based respectively on an affine approximation of the linear programming formulation of the dynamic program, and a Lagrangian relaxation of the dynamic program. Both show theoretically that their formulations give tighter upper bounds on the optimal solution than the DLP bound. We call them the Affine Relaxation bound ($AR$) and the Lagrangian Relaxation bound ($LR$), respectively.

Tight bounds are of great interest as it appears from empirical studies and practical experience that models that give tighter bounds also lead to better controls (better in the sense that they lead to more revenue).

Comparing the AR and LR bounds seems difficult. However, by interposing the PH bound, we are able to clarify the situation about their relative strengths. We show: DLP $\geq$ AR $\geq$ $PHIP$ and $PHLP$ $\geq$ LR $\geq$ Optimal Value. For important classes of networks such as hotel networks or hub-and-spoke networks (with a bank of incoming and outgoing flights), $PHIP = PHLP$ so we get a linear ordering.

# 2 Notation, Demand Model, Controls

## 2.1 Notation

The underlying network will have $m$ resources and $n$ products. The current capacity on resource $i$ is $r_i$ and the vector of capacities $r$ (and $r_t$ to specify time the period). A product is a specification of a combination of resources and a price. (For example an itinerary-fare class combination for an airline network.) We will assume that the booking horizon begins at time 0 and all the resources are consumed instantaneously at time $\tau$. Time is discrete and assumed to consist of $\tau$ intervals, indexed by $t$. We will make the standard assumption that the intervals are fine enough so that at most one customer arrives in each period.

We will denote a product as $j$ and its price as $f_j$ (and the vector of prices as $\mathbf{f}$). A resource $i$ is said to be in product $j$ ($i \in j$) if $j$ uses that resource. We will represent this by $a_{ij} = 1$, and $a_{ij} = 0$ if $i \notin j$, or alternately with the 0-1 matrix $A$. An arrival is a purchase request for a product in a specific interval of time.

## 2.2 Demand Model

The future demand-to-come for each product is a random variable with a known distribution. We will assume that the demands for the different products are independent of each other, independent across time and across products. In period $t$, a request for product $j$ appears with probability $p_{t,j}$. Our assumption of at most one arrival per unit time translates to $\sum_j p_{t,j} \ll 1$. The probability model is that in each period we have a multinomial distribution with parameters $\{p_{t,j}\}$ independently of other periods. Let $p_t$ be the vector of probabilities of products in a period $t$.

The customer behavior assumption is simple: customer who wants product $j$, if it is unavailable, will simply not purchase and disappear (rather than purchasing another product). Together, these assumptions go under the name of the *independent class assumption* in the RM community.

We will be discussing simulation-based methods in this paper. The idea is that we will simulate the future using our forecasts—the implicit assumption being that demand does in fact will follow our assumptions and, statistically, the forecasts. When put this way, it might seem a stretch, but is actually no different from optimizing the expected value given certain distributional assumptions.

An *instance* is a single set of simulated realizations of all the demands for all the products. A large number of instances are generated to capture the variance in the distribution, and each instance-specific data will be represented with a superscript $(k)$. For example: the demand for product $j$ in time period $t$ generated in instance $k$ would be $p^{(k)}_{t,j}$.

### 2.3 Dynamic Program

The dynamic program to determine optimal decision $u^*(t, r)$ (the incidence vector of the acceptable products) for a given capacity vector $r$ is given by the following stochastic dynamic program. Let $\mathcal{U}_r$ be the set of all feasible acceptable products, that is $j \in \mathcal{U}_r$ if $a_{i,j} \leq r_i, \forall i \in j$.

Let $V_t(r)$ denote the maximum expected revenue to go, given remaining capacity $r$ in period $t$ . Then $V_t(r)$ must satisfy the Bellman equation

$$V_t(r) = E\left[\max_{u \in \mathcal{U}_r} \left\{p_t^\top u(t, r) + V_{t+1}(r - Au)\right\}\right] \tag{1}$$

with the boundary condition

$$V_{\tau+1}(r) = 0,\ \forall r. \tag{2}$$

Let $V^*$ denote the optimal value of this dynamic program from 0 to $\tau$ (for a given initial capacity vector).

# 3 Classical bounds

One of the earliest methods proposed for generating network bid-prices is a simple and compact linear program that generally goes by the name of Deterministic Linear Program ($DLP$). See Chapter 3.3.1 of [12] for a more elaborate discussion of this method and its variants. The method consists in solving the following linear program

$$\begin{aligned} & \max && \sum_{t,j} f_j y_{t,j} && (3) \\ (DLP) \quad & \text{s.t.} && \sum_t \sum_{j \ni i} a_{i,j} y_{t,j} \leq r_i,\ \forall i \\ & && 0 \leq y_{t,j} \leq p_{t,j}, \end{aligned}$$

and using the dual prices as the bid-prices for control.

$DLP$ is quite popular as it is very easy to program and can be solved quickly using any off-the-shelf LP software package. Its performance is quite reasonable, and often serves as the benchmark method in simulation comparisons. Its main drawback is that it ignores the variance in the demand.

Consider a simulation where we generate the $N$ instances from the data. The idea is to solve a large number of times and thus hope to capture the variation in the demand. Each such generated instance leads to a perfect-hindsight LP. Let $p_{t,j}^{(k)}$ be the random variable equal to 1 with probability 1 if there is an arrival of $j$ in period $t$ in instance $k$ and 0 otherwise.

$$
(PHLP^{(k)}) \quad \begin{aligned} \max \quad & \sum_{t,j} f_j y_{t,j} \\ \text{s.t.} \quad & \sum_t \sum_{j \ni i} a_{i,j} y_{t,j} \leq r_i, \ \forall i \\ & 0 \leq y_{t,j} \leq p_{t,j}^{(k)}, \end{aligned} \tag{4}
$$

Let $PHIP^{(k)}$ be the problem $PHLP^{(k)}$ with the added restriction that the variables are integer. $PHIP$ is the average of the solution values of $(PHIP^{(k)})$ , and $PHLP$ is the average of the linear programs $(PHLP^{(k)})$.

We do not specify the choice of the number of generated sample paths, $N$, but just assume it is large enough and equal to $n_1 \times n_2 \times \cdots \times n_t \times \cdots \times n_\tau$, where $n_t$ are samples of arrivals drawn in period $t$. An alternate interpretation is that the PH bound calculates $E[\max(\cdot)]$ by simulation.

The fact that these are upper bounds on the optimal solutions are quite easy to prove [13].

## 3.1 New bounds due to Adelman and Topaloglu

Recently Adelman [1] and Topaloglu [14] have proposed new bounds for the network RM problem based respectively on the linear-programming approach to dynamic programming, and the Lagrangian relaxation approach to dynamic programming.

### 3.1.1 AR bound

Let $\mathcal{R}_t$ be the set of state vectors (capacity vectors) at time $t$, and for a given capacity vector $r \in \mathcal{R}_t$, let $\mathcal{U}_r$ be the set of feasible control vectors (that is $a_{i,j} u_j \leq r_i, \forall i \in j$). The formulation for Adelman's $AR$ bound is the following linear program:

$$
\begin{aligned} (AR) \quad \max \quad & \sum_{t, r \in \mathcal{R}_t, u \in \mathcal{U}_r} (\sum_j p_{t,j} f_j u_j) X_{t,r,u} \\ \text{s.t.} \quad & \sum_{r,u} r_i X_{t,r,u} = \sum_{r \in \mathcal{R}_{t-1}, u \in \mathcal{U}_r} (r_i - \sum_j p_{t-1,j} a_{i,j} u_j) X_{t-1,r,u}, \ \forall i, t = 2, \ldots, \tau \\ & \sum_{r \in \mathcal{R}_t, u \in \mathcal{U}_r} X_{t,r,u} = 1, \ \forall t \\ & \sum_{r \in \mathcal{R}_t, u \in \mathcal{U}_r} X \geq 0, \end{aligned} \tag{5}
$$

Adelman [1] shows: $V_{DLP} \geq AR \geq V^*$.

The motivation for this linear program is quite lengthy and, not to unnecessarily repeat here, we refer the reader to the published paper [1] for it.

### 3.1.2 LR bound

For Topaloglu's formulation, assume (without loss of generality) that (i) there is a dummy product $\jmath$ which does not consume any resources and $f_\jmath = 0$ and $p_{t,\jmath} = 1 - \sum_j p_{t,j}$; so from now we just assume that for each $t$, $\sum p_{t,j} = 1$. (ii) We augment the set of legs by a dummy resource $\imath$ with infinite capacity, and all products use one unit of this dummy resource $\imath$.

Let $y_{t,i,j} = 1$ if we accept a request for product $j$ on resource $i$ in period $t$ and 0 otherwise. The optimality condition is

$$\begin{aligned} V_t(r_t) = \max \quad & \sum_j p_{t,j}\{f_j y_{t,\imath,j} + V_{t+1}(r_t - \sum_i y_{t,i,j} a_{i,j} e_i)\} \\ \text{s.t.} \quad & a_{i,j} y_{t,i,j} \le r_{it}, \quad \forall i,j \\ & y_{t,i,j} - y_{t,\imath,j} = 0, \quad \forall t \\ & y_{t,i,j} \in \{0,1\}. \end{aligned} \tag{6}$$

Now relax the constraints of the form $y_{t,i,j} - y_{t,\imath,j} = 0$ with a set of Lagrange multipliers $\lambda = \{\lambda_{t,i,j}\}$ to break it up into leg-level dynamic programs:

$$\begin{aligned} V_t^\lambda(r_t) = \max \quad & \sum_j p_{t,j}\{[f_j - \sum_i \lambda_{t,i,j}] y_{t,\imath,j} + \sum_i \lambda_{t,i,j} y_{t,i,j} + V_{t+1}^\lambda(r_t - \sum_i y_{t,i,j} a_{i,j} e_i)\} \\ \text{s.t.} \quad & a_{i,j} y_{t,i,j} \le r_{t,i} \forall i,j \\ & y_{t,\imath,j}, y_{t,i,j} \in \{0,1\}. \end{aligned} \tag{7}$$

Define the $LR$ bound as $LR = \min_\lambda V_t^\lambda$.

Topaloglu [14] in turn shows: $V_{DLP} \ge LR \ge V^*$.

A direct comparison of Adelman's bound and Topaloglu's bound seems difficult, but interposing the perfect hind-sight bound in between allows us to get a better picture of which one is tighter.

# 4 PHIP vs. AR

In this section we show that the AR bound is weaker than $PHIP$.

**Proposition 1** *$AR \ge PHIP$.*

<u>Proof</u>
At time $t$ for instance $k$, we consider the control in the PH bound as the set of all fare products that we would accept knowing the arrivals from $t+1$ onwards. This therefore

depends on the generated set of arrivals from time $t+1,\ldots,\tau$ in instance $k$ only. Let $u_t^{(k)}$ denote this set. Since we are considering $PHIP$, $u_t^{(k)}$ is well defined and is a set. Define $X_{t,r,u}^{(k)} = 1$ for $u = u_t^{(k)}$ and 0 for all other $u$ assuming that in instance $k$ at time $t$ the remaining capacity vector was $r$. We take the average across the $N$ generated instances, $\bar{X}_{t,r,u} = X_{t,r,u}^{(k)}/N$ and show it to be a feasible solution to the formulation (AR) (alternately, expectation over the sample paths).

First we show that the objective value given by $X_{t,r,u}$ is the same as $PHIP$. Consider the revenue that we obtain in period $t$. Fix a state $r$, and consider the set of all instances which are in state $r$ at time $t$ in the $PHIP$ solution. Every generated instance $k$ is in exactly one state in the optimal $PHIP$ solution, so the states partition all the instances. Conditional on being in state $r$, fix a control set $u$ and consider the instances which are in state $r$ at time $t$ *and* have $u$ as the optimal control set. Again, since we are defining the optimal sets as the maximal sets, each instance belongs to at most one $r$, $u$ combination at time $t$. Conditional on $r$ and $u$ at time $t$, the expected revenue depends now on whether there was an arrival of a product in the set $u$. Hence the expected revenue collected in period $t$ across all the instances is given by

$$\sum_k \sum_{r,u} (\sum_{j\in u} p_{t,j} f_j u_j) X_{t,r,u}^{(k)}.$$

Averaging over all $N$, gives the revenue collected by $PHIP$ as

$$\sum_k \sum_{r,u} (\sum_{j\in u} p_{t,j} f_j u_j) \bar{X}_{t,r,u}.$$

So, the objective value in period $t$ coincides with the objective value of $(AR)$.

Using again the fact that $X_{t,r,u}^{(k)}$ is independent of the arrival in period $t$, we show that $X_{t,r,u}$ satisfies the constraints of $(AR)$.

Let $r_t^{(k)}$ be the state in period $k$ in instance $k$, following the optimal solution given by $PHIP$.

In instance $k$, the variables satisfy the flow-balance constraints for resource $i$

$$r_{i,t}^{(k)} = r_{t-1}^{(k)} - \sum_{j\in u=u_{t-1}^{(k)}} a_{i,j} u_j p_{t-1,j}^{(k)} X_{t-1,r_{t-1}^{(k)},u}^{(k)}.$$

For a given $r_t^{(k)}$, $X_{t,r_t^{(k)},u}^{(k)} = 1$ for $u = u_t^{(k)}$, so

$$r_{i,t}^{(k)} X_{t,r_t^{(k)},u}^{(k)} = r_{t-1}^{(k)} - \sum_{j\in u=u_{t-1}^{(k)}} a_{i,j} u_j p_{t-1,j}^{(k)} X_{t-1,r_{t-1}^{(k)},u}^{(k)}.$$

Define $\mathcal{R}_t|r_{t-1}$ as the set of states achievable at time $t$ given we were in state $r_{t-1}$ at time $t-1$ and we apply some feasible control. Now given we are in $r_{t-1}$ fix a control $u_{t-1}$

and consider the sample paths for which this is the optimal control . By definition this $u_{t-1}$ depends only the arrivals from $t$ till $\tau$, and not on the arrivals in period $t-1$. Taking the expectation by conditioning on $u_{t-1}$ being the optimal set at $t-1$ (given the capacity vector is $r_{t-1}$), and another expectation over arrivals in period $t-1$:

$$\sum_{r\in\mathcal{R}_t|r_{t-1},u\in\mathcal{U}_r} r_i X_{t,r,u} = \sum_{u\in\mathcal{U}_{r_{t-1}}} (r_i - \sum_{j\in u} a_{i,j} u_j p_{t-1,j}) X_{t-1,r_{t-1},u}.$$

Given we are in state $r_{t-1}$ in an instance $k$, the control with perfect hindsight only depends on the arrivals from $t-1$ till $\tau$ and not on how we got to that state. Taking another expectation over all sample paths from period 1 till $t-1$ (alternately, over all the states at $t-1$),

$$\sum_{r\in\mathcal{R}_t,u\in\mathcal{U}_r} r_i X_{t,r,u} = \sum_{\mathcal{R}_{t-1},u\in\mathcal{U}_{r_{t-1}}} (r_i - \sum_{j\in u} a_{i,j} u_j p_{t-1,j}) X_{t-1,r_{t-1},u}.$$

□

# 5 PHLP vs. LR

In this section we show that Topaloglu's Lagrangian bound is tighter than $PHLP$.

The following result is easy to prove, using either a perturbation argument, or observing that the variables $y_{\imath,j,t}$ in (6) can be considered unrestricted.

**Proposition 2** *We can assume that in every period $t$, the Lagrangian multipliers satisfy $\sum_{i\in j} \lambda_{ijt} = f_j$.*

**Proposition 3** *$PHLP \geq LR$.*

Proof
We construct a set of Lagrange multipliers and show that the LR formulation with these multipliers has a value less than or equal to the $PHLP$. As LR minimizes over all Lagrangian multipliers, this proves the result.

Instance $k$ of the PH bound, can be considered a multi-period network RM problem in its own right, and consequently we can generate a set of optimal Lagrangian multipliers $\lambda_{ijt}^{(k)}$. The arrival probabilities in this case are $p_{t,j}^{(k)}$, equal to 1 if there is an arrival of $j$ in period $t$ in this instance $k$ and 0 otherwise. If $p_{t,j}^{(k)} = 0$, we take $\lambda_{ijt}^{(k)} = 0$. Alternately, we consider the Lagrange multipliers as equal to $p_{t,j}^{(k)} \lambda_{ijt}^{(k)}$. If $p_{t,j}^{(k)} = 1$, then the $\lambda_{ijt}^{(k)}$ satisfy $\sum_{i\in j} \lambda_{ijt}^{(k)} = f_j$.

$PHLP^k$ is the DLP formulation for instance $k$, and hence is weaker than the the Lagrangian relaxation for instance $k$, $PH_{LR}^k$ given by:

$$
\begin{aligned}
V_t^{\lambda^{(k)}}(r_t) = \min_{\lambda^{(k)} \mid \sum_{i\in j} \lambda^{(k)}_{t,i,j} = f_j} \max \quad & \sum_j p^{(k)}_{t,j} \{ \sum_i \lambda^{(k)}_{t,i,j} y^{(k)}_{t,i,j} + V_{t+1}^{\lambda^{(k)}}(r_t - \sum_i y^{(k)}_{t,i,j} a_{i,j} e_i) \} \\
\text{s.t.} \quad & a_{i,j} y^{(k)}_{t,i,j} \le r_{it} \forall i, j \\
& y_{i,j,t}, y^{(k)}_{t,i,j} \in \{0,1\}.
\end{aligned} \tag{8}
$$

Define the perfect hindsight Lagrangian bound as $PH_{LR} = \frac{1}{N} \sum_k V_t^{\lambda^{(k)}}(r_t)$.

So, $PHLP \ge PH_{LR}$.

Now consider the $PH_{LR}$ (treating it as a single dynamic program combining all the $PH_{LR}^k$ ) with the additional set of constraints

$$y^{(k)}_{t,i,j} = y^{(l)}_{t,i,j}, \forall k, l, \forall j, t, \text{ and } \forall i \in j.$$

Since we are adding restrictions to a maximization problem, the value of the solution is clearly lower than $PH_{LR}$. Replace all $y^{(k)}_{t,i,j}$ by a common $y_{t,i,j}$. The dynamic program of the Lagrangian relaxation for each instance $k$ is exactly the same as (7) except that instead of $p_{t,j}$, we have $p^{(k)}_{t,j}$.

Of the $N$ instances, let $n_{jt}$ be the number of arrivals of product $j$ in period $t$. Let $\bar{\lambda}_{ijt} = \sum_{k=1}^N p^{(k)}_{t,j} \lambda^{(k)}_{ijt} / n_{jt}$. Notice that $\sum_{i\in j} \bar{\lambda}_{ijt} = f_j$.

By induction, assume the dynamic programs for every state from period $t+1$ onwards (with the added constraints $y^{(k)}_{t,i,j} = y^{(l)}_{t,i,j}$) reduce to $V_{t+1}^{\bar{\lambda}}(\cdot)$.

In period $t$, the co-efficient of $y_{t,i,j}$ is $\frac{n_{jt}}{N} \sum_{k=1}^N p^{(k)}_{t,j} \lambda^{(k)}_{ijt} / n_{jt} = p_{t,j} \bar{\lambda}_{ijt}$. The co-efficient of $V_{t+1}^{\lambda^{(k)}}(\cdot)$ in instance $k$ is $p^{(k)}_{t,j}$. In the problem combining all the $N$ instances, the co-efficient of $V_{t+1}^{\bar{\lambda}}(\cdot)$ is given by $\sum_{k=1}^N \frac{p^{(k)}_{t,j}}{N} = p_{t,j}$.

So we have an instance of a Lagrangian relaxation with a specific set of multipliers $\bar{\lambda}_{ijt}$, it proves the proposition. □

Comments: While we used Proposition 2 in the proof to ignore the term $[f_j - \sum_i \lambda_{t,i,j}]$ in (7), it is not strictly necessary. Note that the operator $[x]^+ = \max\{0, x\}$ is convex, so we can use Proposition 1 of [14] to show that the $PH_{LR}$ is greater than or equal to the Lagrangian bound with $\bar{\lambda}$.

## 6 Some further comments

There is a gap between $PHIP$ and $PHLP$ so we do not quite have a straight ordering of the bounds. However notice that in the case of important classes of networks such as hub-and-spoke networks (set of incoming and outgoing arcs at a hub), or hotel line networks, the matrices for every realization of demand in $PHLP$ are totally unimodular, and we do get $PHLP = PHIP$.

For general networks, it is quite possible that there are instances where Adelman's affine relaxation bound is stronger. It is an open question. It seems unlikely that one can prove that Topaloglu's Lagrangian bound is tighter than $PHIP$ in all cases - consider a deterministic case for a network where integrality does not hold, then the Lagrangian bound is an upper bound on the optimal integer solution.

We chose to analyze these two bounds in this paper, because computational results of Topaloglu of Adelman show that them to be very promising. There are many others methods proposed, notable of which are other Lagrangian based relaxations proposed in Topaloglu and Kunnumkal [15]. They show their bounds to be weaker than the LR bound, but it is not clear how they compare with the PH bounds.

RLP has the advantage that with almost the same computational times as DLP, gives significantly better bounds and bid-prices. We explore improving on it (albeit losing the computational advantages) in related research.

For a given set of time and itinerary specific Lagrangian multipliers $\lambda_{ijt}$, once we decompose to leg-level problems, one can then take a linear programming formulation of the leg-level dynamic program (as in Adelman [1]) and observe that all the $\lambda_{ijt}$'s can be moved to the right hand side of the linear programs - this gives an alternate proof of the convexity of the Lagrangian over the $\lambda_{ijt}$'s.

Using this fact, one can strengthen the Lagrangian relaxation for a given set of $\lambda_{ijt}$'s by first relaxing and adding aggregate constraints across the linear programs – this still preserves convexity and strengthens the bound. We pursue this line of research with computational results in a related paper.

The results of this paper can be extended to a choice model of customer behavior for a network with a model similar to the ones in (Talluri and van Ryzin [11],Talluri [9], van Ryzin and Liu [16], Gallego, Iyengar, Phillips and Dubey [5]).

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