A mathematical excursion: from the three–door problem to a Cantor–type perfect set

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1 Introduction

We invite you, reader, on a mathematical trip. Our starting point will be a well–known problem, the three–door problem; our endeavours to solve it will take us to a beautiful representation system for the real numbers in $(0,1]$ which, in turn, will provide us with an actually computable enumeration of the positive rationals (in the sense that we shall be able to tell what rational is in position n and, viceversa, given a rational find its position in the enumeration); as a bonus we shall easily prove the irrationality of e. Lastly, already exhausted, our trip will end in the dark region of mysterious sets, finding a simple description of a Cantor–type perfect set contained in $[0,1]$.

2 Starting point: the three–door problem.

Mathematics have always been enriched by a diversity of games and intellectual curiosities which have provided, throughout its history, an endless supply of problems which have acquired a life of their own, far removed from the recreational aspect of their origins. In this way the first building blocks of probability owe their existence to the analysis of gambling games carried out by Fermat and Pascal back in the beginning of the XVIIth century. Undoubtedly Fermat himself was much attracted to mathematics thanks to Bachet's *Problèmes plaisants et délectables* of 1612, [1], which was an introduction to Bachet's most famous book: the latin translation of Diophantus' *Arithmetica*, in whose margins Fermat wrote the note that made his major theorem famous. Lastly, to mention another important instance, E. Lucas' *Récréations mathématiques*, [13], constituted a source of interesting problems in the beginning of the present century.

In our case, let us start our excursion with the setting of a "simple" problem which is presented in the form of a quite innocent game.

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2.1 The three–door problem.

In a TV contest, one of three shut doors hide a wonderful prize while the other two open onto a dismal void.... The host proposes that the contestant choose one of the three doors. After choosing, to make things a bit more interesting, the host opens one of the other two doors wide showing the absence of any prize and offers the constestant the possibility of changing his/her choice. Our hero/heroine (the contestant, obviously!) has to take a dramatic decision: to change or not to change.

Our mathematical challenge is to help the contestant to make his/her decision finding the probability of both possibilities. A (widely accepted) solution to the problem is assigning probability 1/3 to the conservative option (not to change) and 2/3 to the daring option (to change). One of the easiest ways of reaching this conclusion is the following reasoning:

> *The probability of choosing the right door in the first place is unquestionably* 1/3. *The probability that one of the other doors contain the prize is then,* 2/3. *If we choose not to change when we are offered the chance, our door still has the same probability of success,* 1/3 *while now, the other door accumulates a probability of* 2/3 *of containing the prize. Without hesitation, we must change.*

2.2 A re–formulation of the problem.

We suggest our readers tackle the same problem with a slightly different setting: the n–door–problem.

> In a TV contest, only one of n closed doors hides a prize. A contestant randomly chooses one of the doors and then the host opens one of the other $n-1$ doors without a prize. The contestant is offered the chance of changing the previous choice. In case of changing, the old door gets completely mixed with the others becoming indistinguishable. After the new choice is made, the host opens another of the $n-2$ doors left that hides nothing and offers the contestant the possibility of changing, and so on till only two doors remain closed, i.e. the last choice of our contestant and the door that the host has not opened.
>
> A mathematician, having followed the whole process attentively, says: 'I have followed the strategy of the contestant and I can say that his/her probability of winning is 11/42'.
>
> A second mathematician, who has been fast asleep during the whole contest (not even knowing the value of n), wakes up and hears the last utterance saying: 'From what my colleague says, there were 7 doors and the contestant changed on two occasions, when there were 4 and 3 doors to choose from'.
>
> Can you explain how the two mathematicians reached their conclusions?

We suggest our reader takes a rest and spends some time trying to find the probability of each one of the 2^{n-2} possible strategies that our contestant can follow. A strategy can be represented by a strictly decreasing sequence of positive integers $\{n, a_k, a_{k-1}, \ldots, a_2, a_1\}$ such that

$$n > a_k > a_{k-1} > \ldots > a_2 > a_1 \geq 1.$$

where a_i denotes that a change of doors was made when there were a_i doors to choose from (notice that $a_k \neq n-1$).

2.3 The solution to the problem.

If no change whatsoever is made, the probability of winning is obviously $1/n$. If a last minute change is made (when there is only one door offered besides the initially chosen), the probability of winning will be $(n-1)/n$.

If we describe any other strategy by the above convention, our first change is made when we can choose from a_k doors. The probability of choosing the right door will be the probability of having previously chosen the wrong door times the probability of choosing correctly among a_k doors, that is:

$$p_k = \left(1 - \frac{1}{n}\right)\frac{1}{a_k}.$$

For our next change, reasoning in the same way we would have,

$$p_{k-1} = (1-p_k)\frac{1}{a_{k-1}} = \left(1 - \left(1 - \frac{1}{n}\right)\frac{1}{a_k}\right)\frac{1}{a_{k-1}},$$

which can be expressed

$$p_{k-1} = \frac{1}{a_{k-1}} - \frac{1}{a_{k-1}\cdot a_k} + \frac{1}{a_{k-1}\cdot a_k \cdot n}.$$

Iterating the process we have for the last change

$$p_1 = (1-p_2)\frac{1}{a_1} = \frac{1}{a_1} - \frac{1}{a_1\cdot a_2} + \cdots + \frac{(-1)^{k-1}}{a_1\cdot a_2\cdots a_k} + \frac{(-1)^k}{a_1\cdot a_2\cdots a_k\cdot n}. \quad (1)$$

In expression (1) we obviously have

$$1 \leq a_1 < a_2 < \ldots < a_k < n-1.$$

A strategy gets completely described by a subset of the set $\{1, 2, \ldots, n-2\}$; ($\emptyset$ corresponds to the strategy of making no change at all). For each strategy, $\{a_1, a_2, \ldots, a_k\} \subset \{1, 2, \ldots, n-2\}$, the probability of winning is expression (1).

This accounts for our first mathematician's assertion as

$$n = 7, k = 2, a_2 = 4, a_1 = 3 \quad \text{and} \quad p_1 = \frac{1}{3} - \frac{1}{3\cdot 4} + \frac{1}{3\cdot 4\cdot 7} = \frac{11}{42}.$$

To account for our second (and quite smart) mathematician's claim, we can play a little with what we have and find a few more probabilities in the case $n = 7$. We could end up with a table similar to the following:

Strategy	Probability
$\{1,2,3,4,5\}$	$1-\frac{1}{2}+\frac{1}{2\cdot 3}-\frac{1}{2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}-\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 7}=\frac{62}{105}$
$\{1,2,3,4\}$	$1-\frac{1}{2}+\frac{1}{2\cdot 3}-\frac{1}{2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 7}=\frac{53}{84}$
$\{1,2,3\}$	$1-\frac{1}{2}+\frac{1}{2\cdot 3}-\frac{1}{2\cdot 3\cdot 7}=\frac{9}{14}$
$\{2,4,5\}$	$\frac{1}{2}-\frac{1}{2\cdot 4}+\frac{1}{2\cdot 4\cdot 5}-\frac{1}{2\cdot 4\cdot 5\cdot 7}=\frac{111}{280}$
...	...

A patient completion of the former table (2^5 entries) would show a most interesting fact: to different strategies there correspond different probabilities. This motivates the following result.

Theorem 2.1 *Any rational $p/q \in (0,1]$ expands uniquely in the following way:*

$$\frac{p}{q}=\frac{1}{a_1}-\frac{1}{a_1\cdot a_2}+\ldots+\frac{(-1)^{k-1}}{a_1\cdot a_2\cdots a_k}\,, \qquad (2)$$

where a_i are positive integers verifying:

$$1\le a_1<a_2<\ldots<a_{k-1}<a_k-1.$$

PROOF The interval $(0,1]$ can be expressed as the disjoint reunion

$$(0,1]=\bigcup_{n=1}^{\infty}\left(\frac{1}{n+1},\frac{1}{n}\right].$$

Now, given a $\alpha\in(0,1]$, α will belong to one of the intervals

$$\left(\frac{1}{n+1},\frac{1}{n}\right].$$

Consequently,

$$\alpha=\frac{1}{n}-\lambda_1\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{n}-\frac{\lambda_1}{n(n+1)},$$

for a $\lambda_1\in[0,1)$. If we denote $\alpha_1=\lambda_1/(n+1)$, we have the equality,

$$\alpha=\frac{1}{n}-\frac{1}{n}\cdot\alpha_1$$

and $\alpha_1 \in (0, \frac{1}{n+1})$. Applying the same procedure to α_1 we get

$$\alpha_1 = \frac{1}{m} - \frac{1}{m} \cdot \alpha_2 \qquad (m > n)$$

and therefore we will eventually get an expansion of the form (2):

$$\alpha = \frac{1}{n} - \frac{1}{n \cdot m} \cdot \alpha_2 \qquad (m > n)$$

In point of fact, the algorithm that leads to (2) can be summarized by iterating the two operations

$$\begin{aligned} a_i &= \left\lfloor \frac{1}{\alpha_{i-1}} \right\rfloor \qquad \text{with} \quad \alpha_0 = \alpha, \\ \alpha_i &= 1 - \alpha_{i-1} \cdot a_i. \end{aligned} \tag{3}$$

($\lfloor x \rfloor$ denotes the greatest integer less or equal than x.)

If α is irrational, all the α_i will be irrational and the algorithm will never terminate (this proves more than promised in the phrasing of the theorem; it proves the existence of an infinite expansion of the form (2) for any irrational in $(0, 1]$).

If $\alpha = p/q$ is a rational number in lowest terms, the algorithm is easily seen to become a modified Euclidian algorithm. If we divide q by p :

$$q = a_1 \cdot p + r_1 \qquad (0 \le r_1 < p),$$

it is obvious that

$$\left\lfloor \frac{q}{p} \right\rfloor = a_1 \qquad \text{and} \qquad \frac{r_1}{q} = \alpha_1.$$

Next we would perform the division of q by r_1 :

$$q = a_2 \cdot r_1 + r_2 \qquad (0 \le r_2 < r_1),$$

and so on. As the sequence of remainders r_i will be strictly decreasing $p > r_1 > r_2 > \dots$, the algorithm will eventually terminate with $r_k = 0$. Therefore the expansion (2) will be finite. The last two divisions being

$$\begin{aligned} q &= a_{k-1} \cdot r_{k-2} + r_{k-1} \qquad (0 \le r_{k-1} < r_{k-2}) \\ q &= a_k \cdot r_{k-1}, \end{aligned}$$

will imply

$$a_{k-1} \cdot r_{k-2} = (a_k - 1) \cdot r_{k-1}$$

and as $r_{k-1} < r_{k-2}$ we will have $a_{k-1} < a_k - 1$.

The uniqueness of the expansions comes from the double inequality

$$\frac{1}{a_1 + 1} < \frac{1}{a_1} - \frac{1}{a_1 \cdot a_2} + \dots \le \frac{1}{a_1}.$$

The only duplicate expansion is obtained in the finite case, due to the equality

$$\frac{1}{n+1} = \frac{1}{n} - \frac{1}{n(n+1)}.$$

That is the reason for the exclusion of two consecutive integers at the end of the expansion. □

We will denote the expansion (2) by

$$\langle a_1, a_2, \ldots \rangle.$$

Now, we see how our second mathematician performed his/her trick. The probability mentioned by his/her colleague was 11/42:

$$\begin{aligned} 42 &= 11 \cdot \mathbf{3} + 9 \\ 42 &= 9 \cdot \mathbf{4} + 6 \\ 42 &= 6 \cdot \mathbf{7}. \end{aligned}$$

Thus

$$\frac{11}{42} = \frac{1}{3} - \frac{1}{3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 7} = \langle 3, 4, 7 \rangle.$$

The uniqueness of the expansion allows our clever friend to say:

> *there were 7 doors and the contestant changed on two occasions, when there were 4 and 3 doors to choose from.*

3 A representation system for the real numbers in $(0, 1]$.

We get to the first resting spot of our excursion. Looking backwards we see that we have solved our generalised n–door problem and, at the same time, we have discovered a system of representation for the real numbers, α in $(0, 1]$:

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \cdots$$

where $1 \leq a_1 < a_2 < a_3 < \ldots$.

The first mathematicians who paid any attention to these expansions were Lambert and Lagrange, see [10, 11]. Posteriorly, Ostrogradsky, see [17], and Sierpiński, [21], were the first to develop a few of their numerical properties. Pierce in 1929, see [16], used the model in an algorithm to find algebraic roots of polynomials. Some authors have used Pierce's name to denote the expansion which had been previously refered to as "Lambert fractions" or "ascending fractions". The most interesting modern presentation can be found in Shallit (1986), see [19], who developed at the same time the metric theory of the model following the methods used for the non-alternated expansions (Engel's series) developed in 1947 by Borel, see [2], and Lévy, see [12], and later by Erdös, Rényi and Szüsz in their paper [3] of 1958, improved by Rény in 1962, see [18]. There is also a 1987 paper by A. Knopfmacher and J. Knopfmacher, [8], who use the model to build $\mathbb{R}$. Some interesting new results related to Pierce expansions can be found in [20] and [9].

The easiest infinite Pierce expansion is $\langle 1, 2, 3, 4, \ldots \rangle$ which coincides with the Taylor expansion of $1 - e^x$ for $x = -1$:

$$1 - \frac{1}{e} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots = \langle 1, 2, 3, \ldots \rangle.$$

Incidentally, this proves the irrationality of e.

Other examples are not so obvious. As a curiosity we mention

$$\langle 1,3,5,7,\ldots\rangle = \frac{1}{\sqrt{e}}\sum_{n=0}^{\infty}\frac{1}{2^n\cdot n!\cdot(2n+1)}.$$

As a system of representation, Pierce expansions are not bad at all. If $\alpha = \langle a_1, a_2, \ldots, a_n, \ldots\rangle$, is the truncation of the development at level n, it provides quite a good approximation to the number represented:

$$|\alpha - \langle a_1, a_2, \ldots, a_n\rangle| < \frac{1}{a_1\cdots a_n\cdot a_{n+1}}, \tag{4}$$

which, in the worst case ($a_i = i,\ i = 1,2,\ldots$), is of the order $1/(n+1)!$.

4 A computable enumeration for Q^+.

There exist different ways of enumerating Q^+, but not one of these known enumerations allow us to actually compute the position of a given rational or, viceversa, find which rational occupies a given position. The most usual enumeration is the diagonal ordering of the rationals:

$$\frac{1}{1},\frac{2}{1},\frac{1}{2},\frac{3}{1},\frac{2}{2},\frac{1}{3},\frac{4}{1},\frac{3}{2},\frac{2}{3},\frac{1}{4},\frac{5}{1},\frac{4}{2},\frac{3}{3},\frac{2}{4},\frac{1}{5},\ldots$$

All fractions appear in the scheme above but duplicated infinitely many times; to find the position of p/q one can compute $(1/2)(p+q-1)(p+q-2)+q$. To determine the position of the irreducible ones is, as Prof. Hardy says in [4], *more complicated*. So, even if it is theoretically possible to reach a given rational number in this enumeration, it is obvious that in practice it cannot be achieved. As we see the problem (you can find more about this and other enumerations in our paper [15]), its nature is intimately related to the representation of rational numbers. The basic idea is using the finite subsets of $\mathbb{N} = \{1,2,3,\ldots\}$ through the binary representation of a positive integer n:

$$n = 2^{b_1} + 2^{b_2} + \cdots + 2^{b_r} \longrightarrow \{b_1+1, b_2+1, \ldots, b_r+1\},$$

where $0 \le b_1 < b_2 < \ldots < b_r$.

From here, the method consists in finding a system of representation for Q^+ such that all positive rationals correspond one–to–one with the finite subsets of $\mathbb{N}$. Pierce expansions provide the system of representation we need.

To any strictly increasing finite sequence of positive integers,

$$\{a_1, a_2, \ldots, a_k\} \qquad \text{with} \qquad 1 \le a_1 < a_2 < \cdots < a_k,$$

we may associate the positive integer:

$$n = 2^{a_1-1} + 2^{a_2-1} + \cdots + 2^{a_k-1}$$

or, what amounts to the same, the number n such that in the binary system is written, from right to left, as 1 in positions $a_1, a_2, \ldots, a_k$ and 0 in the rest of places. For example,

$$\{1,3,5,8\} \longrightarrow 2^0 + 2^2 + 2^4 + 2^7 = 10010101.$$

Now, to any rational number $p/q \in (0,1]$ we can associate its Pierce expansion, $\langle a_1, a_2, \ldots, a_k \rangle$, which may be regarded as the strictly increasing finite sequence of positive integers $\{a_1, a_2, \ldots, a_k\}$ with the particularity that $a_k > 1 + a_{k-1}$. According to the previous assignment its corresponding positive integer would be of the form (binary) $10\ldots$, with a 0 in the last but one position as we go from right to left. To any rational number $q/p > 1$, we associate the Pierce expansion corresponding to its inverse $p/q = \langle a_1, a_2, \ldots, a_k \rangle$ and from this, we consider the strictly increasing finite sequence of positive integers $\{a_1, a_2, \ldots, a_{k-1}, a_k - 1, a_k\}$. Its corresponding positive integer would be of the form $11\ldots$, with a 1 in the last but one position as we go from right to left.

Conversely, to any positive integer n written in the binary system as

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_k} \qquad \text{with} \qquad 0 \leq a_1 < a_2 < \ldots < a_k,$$

we assign the following rational number:

1. If $a_k > 1 + a_{k-1} \longrightarrow \langle 1 + a_1, 1 + a_2, \ldots, 1 + a_k \rangle \in (0,1]$
2. If $a_k = 1 + a_{k-1} \longrightarrow \dfrac{1}{\langle 1 + a_1, 1 + a_2, \ldots, 1 + a_{k-2}, 1 + a_k \rangle} > 1$

The uniqueness of the Pierce expansion of any rational number in $(0,1]$ ensures the bijectivity of the map just defined between $\mathbb{N}$ and Q^+.

5 A closer look to Pierce expansions

Let us rest for a while to contemplate what we have achieved. Quite a bit. As we rest, it is worth our while to examine Pierce expansions from a closer point of view. For each real number in (0,1], we may consider the i–th projection ω_i as the map that assigns to a real number its i–th partial quotient: if $x = \langle a_1, a_2, a_3, \ldots \rangle$, then $\omega_i(x) = a_i$.

We will call a *cylinder* of order k the set of numbers such that the first k partial quotients are fixed:

$$C(a_1, a_2, \ldots, a_k) = \{x \in (0,1] : \omega_1(x) = a_1, \omega_2(x) = a_2, \ldots, \omega_k(x) = a_k\}.$$

It is seen at once that cylinders of any order are intervals of length:

$$|C(a_1, a_2, \ldots, a_k)| = \frac{1}{a_1 \cdot a_2 \cdots a_k \cdot (1 + a_k)}.$$

It is also clear that a cylinder of order k is the disjoint union of all the cylinders of order $k+1$ contained in it:

$$C(a_1, a_2, \ldots, a_k) = \bigcup_{j=1+a_k}^{\infty} C(a_1, a_2, \ldots, a_k, j).$$

Obviously, we can consider generalized cylinders, in which the fixed partial quotients are not the first k. The problem is that these are not intervals any more (though they still are unions of intervals). The most simple is the following:

$$H[\omega_k = n] = \{x \in (0,1] : \omega_k(x) = n\},$$

which is a reunion of cylinders:

$$H[\omega_k = n] = \bigcup_{1 \le a_1 < a_2 < \ldots < a_{k-1} \le n-1} C(a_1, a_2, \ldots, a_{k-1}, n).$$

Consequently, its Lebesgue measure is:

$$\begin{aligned} &\lambda(H[\omega_k = n]) = \\ &= \sum_{1 \le a_1 < a_2 < \ldots < a_{k-1} \le n-1} \frac{1}{a_1 a_2 \cdots a_{k-1} n(n+1)} = \\ &= \frac{1}{n(n+1)} \sum_{1 \le a_1 < a_2 < \ldots < a_{k-1} \le n-1} \frac{1}{a_1 a_2 \cdots a_{k-1}}. \end{aligned}$$

There are different ways of evaluating this last sum. A beautiful one is the following. Multiplying inside by $(n-1)!$ and dividing outside by the same quantity we get:

$$\sum_{1 \le a_1 < \ldots < a_{k-1} \le n-1} \frac{1}{a_1 a_2 \cdots a_{k-1}} = \frac{1}{(n-1)!} \sum_{1 \le a_1 < \ldots < a_{n-k} \le n-1} a_1 a_2 \cdots a_{n-k}. \tag{5}$$

Now, this last sum, can be viewed as the coefficient of x^k in the polynomial

$$x(x+1)(x+2)\cdots(x+n-1). \tag{6}$$

The coefficients of this last polynomial are known as *Stirling numbers of the second kind*. The properties of Stirling numbers (both of the first and the second kind) can be found in Jordan, [6], or, to cite a more recent reference, in Graham, KJnuth & Patashnik, [5], whose notation we follow: $\left[n \atop k \right]$. Thus,

$$x(x+1)(x+2)\cdots(x+n-1) = \left[n \atop 1 \right] x + \left[n \atop 2 \right] x^2 + \cdots + \left[n \atop n \right] x^n. \tag{7}$$

For our purposes, we only need a very simple property of Stirling numbers:

$$\left[n \atop 1 \right] + \left[n \atop 2 \right] + \cdots + \left[n \atop n \right] = n! \tag{8}$$

which is trivially obtained from (7) replacing x by 1.

Now, the last sum in (5) can be written as:

$$\begin{aligned} &\sum_{1 \le a_1 < \ldots < a_{k-1} \le n-1} \frac{1}{a_1 a_2 \cdots a_{k-1}} = \\ &= \frac{1}{(n-1)!} \sum_{1 \le a_1 < \ldots < a_{n-k} \le n-1} a_1 a_2 \cdots a_{n-k} = \\ &= \frac{\left[n \atop k \right]}{(n-1)!}. \end{aligned}$$

and we have, finally:

$$|H(\omega_k = n)| = \frac{1}{n(n+1)} \cdot \frac{\left[n \atop k \right]}{(n-1)!} = \frac{\left[n \atop k \right]}{(n+1)!}.$$

With these results it is now very easy to prove the following:

THEOREM 1. *The set of real numbers in* $(0, 1]$ *such that the integer* n *appears in its Pierce expansion has Lebesgue measure* $1/(n + 1)$.

Theorem 1 has an immediate corollary:

THEOREM 2. *The set of real numbers in* $(0, 1]$ *such that the integer* n *does not appear in its Pierce expansion has Lebesgue measure* $n/(n + 1)$.

It is not difficult to generalize theorem 2:

THEOREM 3. *The set of real numbers in* $(0, 1]$ *such that the integers* n *and* m *do not appear in its Pierce expansion has Lebesgue measure :*

$$\left(1 - \frac{1}{n+1}\right) \cdot \left(1 - \frac{1}{m+1}\right).$$

All these results can be seen in Shallit's paper [19].

6 A Cantor–type perfect set

Let us consider the set, C, of real numbers in $(0, 1]$ such that their Pierce expansion contains no odd integers. According to theorem 3 of the former section, the Lebesgue measure of C is:

$$\lambda(C) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{2n}\right) = 0.$$

The set C has the power of the continuum as we can establish a one–to–one correspondence between its elements and $(0, 1]$:

$$\langle a_1, a_2, \ldots, a_n, \ldots \rangle \leftrightarrow \langle \frac{a_1}{2}, \frac{a_2}{2}, \ldots, \frac{a_n}{2}, \ldots \rangle.$$

It is also easy to prove that C has the same structure as Cantor's set: C is closed and dense–in–itself (all points in C are limit–points), thus a *perfect* set, and C is a frontier set (its complement is a reunion of intervals).

7 A Cantor–type perfect set formed exclusively by trascendental numbers

In 1848 (1851) J. Liouville established a very important result which permitted him to exhibit, for the first time in mathematics, a trascendental real number, that is a real number which is not the root of any polynomial equation with real coefficients. These last numbers are called algebraic of degree n if n is the lowest degree of the polynomials which have it as a root.

LIOUVILLE'S THEOREM. *If α is algebraic of degree n, there exists a constant M which depends only of α such that for all rational number a/b we always have*

$$\left|\alpha - \frac{a}{b}\right| \geq \frac{M}{b^n}.$$

(See [14] for details).

Liouville's theorem states that algebraic numbers are not well approximated by rational numbers in the sense that any rational approximation to α with a given denominator has a closeness "boundary" which cannot be surpassed.

In our case, let us consider the following Pierce expansion of a real number

$$\ell_p = \langle p^{2!}, p^{3!-2!}, \ldots, p^{n!-(n-1)!}, \ldots \rangle = \frac{1}{p^{2!}} - \frac{1}{p^{3!}} + \frac{1}{p^{4!}} - \cdots + \frac{(-1)^n}{p^{n!}} + \cdots, \quad (9)$$

where p is any positive integer.

The number ℓ_p is trascendental because inequality (4) tells us that

$$\left|\ell_p - \left(\frac{1}{p^{2!}} - \frac{1}{p^{3!}} + \cdots + \frac{(-1)^k}{p^{k!}}\right)\right| = \left|\ell_p - \frac{a}{p^{k!}}\right| < \frac{1}{p^{(k+1)!}} < \frac{1}{(p^{k!})^k}, \quad (10)$$

thus contradicting Liouville's theorem if it were algebraic of degree k.

We consider now the set L_p of all real numbers in $(0, 1]$ such that their Pierce expansions contain only integers extracted from the Pierce expansion of ℓ_p. As we did in the previous section, it is seen at once that L_p has measure zero, power of the continuum and as all of them will verify inequalities as (10), all of them are trascendental.

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