

On the goodness of fit of Kirk's formula for spread option prices

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Abstract

In this paper we investigate the goodness of fit of the Kirk's approximation formula for spread option prices in the correlated lognormal framework. Towards this end, we use the Malliavin calculus techniques to find an expression for the short-time implied volatility skew of options with random strikes. In particular, we obtain that this skew is very pronounced in the case of spread options with extremely high correlations, which cannot be reproduced by a constant volatility approximation as in the Kirk's formula. This fact agrees with the empirical evidence. Numerical examples are given.

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JEL code: G12, G13

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1 Introduction

A spread option is a derivative written as the difference of two underlying assets. Namely, the payoff of a call spread option with strike K with time to maturity T is

$$(S_T^1 - S_T^2 - K)_+,$$

where S^1 and S^2 denote two asset prices. It is well-known that if S^1 and S^2 are two geometric Brownian motions (that may be correlated) and $K = 0$, the corresponding option price is given by the Margrabe formula (see Margrabe (1978)), which can be deduced from the fact that S^1/S^2 is a log-normal process. Thus, in this case, the spread option value can be expressed as the classical Black-Scholes call price with initial asset price S_0^1 , where we take the strike

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equal to the expected value of S_T^2 and volatility equal to $\sqrt{\sigma^2 - 2\rho\sigma\sigma' + (\sigma')^2}$. Here σ and σ' are the volatility parameters of S^1 and S^2 , respectively, and ρ denotes the correlation.

In the case $K \neq 0$ the fraction $S^1 / (S^2 + K)$ is not log-normal and then the arguments used in the deduction of Margrabe formula cannot be applied anymore. One proposed solution is to assume the distribution of the spread is able to be approximated by the normal distribution. This leads us to the Bachelier's method (see Shimko (1994)). Even though the approximation of the spread distribution by the normal one is poor, this method can be precise in some ranges of parameters (see, for example, Carmona and Durrleman (2003)). A more successful method is suggested by Kirk (1995), who applied the Margrabe formula, approaching $S^1 / (S^2 + K)$ by a log-normal random process. Nowadays Kirk's formula is the most popular option pricing approximation expression for spread options due to its accuracy and its simplicity. Recently, some other authors have proposed new formulas and methods to estimate spread option prices. Among them, we can mention Alexander and Venkatramanan (2011), Bjerksund and Stensland (2011), Borovkova et al. (2007), Carmona and Durrleman (2003), and Deng et al. (2008). Numerical evidence has shown that these approaches may improve Kirk's estimates, specially for high correlated cases (see, for instance, Bjerksund and Stensland (2011)).

One interesting tool in the development of an accurate approximation formula is the knowledge of the properties of the corresponding implied volatility. For example, in the case of vanilla options with stochastic volatility, we know that implied volatilities exhibits smiles and skews (see Renault and Touzi (1996)). Then, we have to take into account that accurate approximations should reproduce them and, moreover, we can understand why some formulas fail (for example, we expect that a constant volatility expression will fail for in or out-of-the-money options).

Up to our knowledge, there are not similar analytical studies in the case of options with random strikes (as for example spread options). The main goal of this paper is to consider analytically the relation between the implied volatility and the stock price S^1 for options with random strike. By means of the Malliavin calculus techniques, we develop an extension of the Margrabe formula that allows us to find an expression for the short-time skew slope for spread options. This analysis of the implied volatility skew evidences that the dependence between the implied volatility and the stock price S^1 increases strongly when the correlation parameter ρ is close to 1. Hence we can expect that Kirk's approximation reduces its accuracy for highly correlated assets due to the fact that in this approximation the volatility is constant as a function of the stock price S^1 . This agrees with the numerical evidence (see, for example, Baeva (2011)).

The organization of this paper is as follows. In Section 2 we introduce the framework of this paper. Section 3 is devoted to present an extended Margrabe formula for the case $K \neq 0$. This formula is used in Section 4 to figure up an expression for the derivative of the implied volatility with respect to the log-stock price. The short time behaviour of this derivative is analyzed in Section

5. Finally in Section 6 we obtain our results on the goodness of the fit of Kirk's formula as an application.

2 Statement of the model and notation

In this paper we consider the following model for the log-price of a stock under a risk-neutral probability measure Q :

$$dX_t = \left(r - \frac{\sigma_t^2}{2} \right) dt + \sigma_t (\rho dW_t + \sqrt{1 - \rho^2} dB_t), \quad t \in [0, T]. \quad (1)$$

Here, r is the instantaneous interest rate, W and B are independent standard Brownian motions and $\rho \in (-1, 1)$. For the sake of simplicity, we assume that the volatility process σ is a square-integrable deterministic function which is right-continuous. In the following we denote by \mathcal{F}^W and \mathcal{F}^B the filtrations generated by W and B , respectively. We define $\mathcal{F} := \mathcal{F}^W \vee \mathcal{F}^B$.

In this paper we consider European call options with payoff $h(X_T) := (e^{X_T} - K_T)_+$, where we allow the strike K_T to be random. More precisely, we assume K is a square-integrable, positive, continuous, bounded and \mathcal{F}_t^W -measurable process.

Notice that this choice includes some popular classes of options as basket ones. It is well-known that the price of an European call with strike K_T is given by the formula

$$V_t = e^{-r(T-t)} E \left((e^{X_T} - K_T)_+ | \mathcal{F}_t \right). \quad (2)$$

In the sequel, we will make use of the following notation:

- $M_t^T := E [K_T | \mathcal{F}_t^W]$. Observe that, by the martingale representation theorem,

$$M_t^T = E(K_T) + \int_0^t m(T, s) dW_s,$$

for some \mathcal{F}^W -measurable and adapted process $m(T, \cdot)$.

- $v_t := \left(\frac{Y_t}{T-t} \right)^{\frac{1}{2}}$, with $Y_t := \int_t^T a_s^2 ds$, where $a_s^2 ds := \sigma_s^2 ds - 2 \frac{d\langle M^T, X \rangle_s}{M_s^T} + \frac{d\langle M^T, M^T \rangle_s}{(M_s^T)^2}$. Note that

$$\begin{aligned} a_s^2 &= \sigma_s^2 - 2\rho\sigma_s \frac{m(T, s)}{M_s^T} + \frac{m^2(T, s)}{(M_s^T)^2} \\ &= \sigma_s^2 (1 - \rho^2) + \left(\rho\sigma_s - \frac{m(T, s)}{M_s^T} \right)^2 \end{aligned}$$

is a positive quantity. Although the right-hand-side of the last equality depends on T , we denote it by a_s^2 in order to simplify the notation. It is easy to see that for all $\rho \in (-1, 1)$, $a_s^2 \geq C(\rho) \sigma_s^2$.

- $BS(t, x, K, \sigma)$ denotes the price of an European call option under the classical Black-Scholes model with constant volatility σ , current log stock price x , time to maturity $T - t$, strike price K and interest rate r . Remember that in this case:

$$BS(t, x, K, \sigma) = e^x N(d_+) - K e^{-r(T-t)} N(d_-),$$

where N denotes the cumulative probability function of the standard normal law and

$$d_{\pm} := \frac{x - \tilde{x}_t^*}{\sigma \sqrt{T-t}} \pm \frac{\sigma}{2} \sqrt{T-t},$$

with $\tilde{x}_t^* := \ln K - r(T-t)$.

- $\mathcal{L}_{BS}(\sigma^2)$ stands for the Black-Scholes differential operator, in the log variable, with volatility σ :

$$\mathcal{L}_{BS}(\sigma^2) = \partial_t + \frac{1}{2} \sigma^2 \partial_{xx}^2 + (r - \frac{1}{2} \sigma^2) \partial_x - r.$$

It is well known that $\mathcal{L}_{BS}(\sigma^2) BS(\cdot, \cdot, \sigma) = 0$.

Now we describe some basic notation that is used in this article. For this, we assume that the reader is familiar with the elementary results of the Malliavin calculus, as given for instance in Nualart (2006).

Let us consider a standard Brownian motion $Z = \{Z_t, t \in [0, T]\}$ defined on a complete probability space (Ω, \mathcal{F}, P) . The set $\mathbb{D}_Z^{1,2}$ is the domain of the derivative operator D^Z in the Malliavin calculus sense. $\mathbb{D}_Z^{1,2}$ is a dense subset of $L^2(\Omega)$ and D^Z is a closed and unbounded operator from $L^2(\Omega)$ into $L^2([0, T] \times \Omega)$. We also consider the iterated derivatives $D^{Z,n}$, for $n > 1$, whose domains is denoted by $\mathbb{D}_W^{n,2}$.

The adjoint of the derivative operator D^Z , denoted by δ^Z , is an extension of the Itô integral in the sense that the set $L_a^2([0, T] \times \Omega)$ of square integrable and adapted processes (with respect to the filtration generated by Z) is included in $\text{Dom} \delta^Z$ and the operator δ^Z restricted to $L_a^2([0, T] \times \Omega)$ coincides with the Itô integral. We make use of the notation $\delta^Z(u) = \int_0^T u_t dZ_t$. We recall that $\mathbb{L}^{n,2} := L^2([0, T]; \mathbb{D}_W^{n,2})$ is included in the domain of δ^Z for all $n \geq 1$.

3 A decomposition result

Before proving an extension of the Hull and White formula, we state the following result, which is needed in the remaining of the paper.

Lemma 1 Let K_T be bounded, $0 \leq t \leq s < T$ and $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_T^W$. Assume $\rho \in (-1, 1)$. Then, for any $n \geq 0$, there exists $C = C(n, \rho)$ such that

$$\begin{aligned} & |E((\partial_x^{n+2} - \partial_x^{n+1})BS(s, X_s, M_s^T, v_s)|\mathcal{G}_t)| \\ & \leq C \left(\int_t^T \sigma_\theta^2 d\theta \right)^{-\frac{1}{2}(n+1)}. \end{aligned}$$

Proof: In order to show this result, we proceed as in the proof of Lemma 4.1 in Alòs, León and Vives (2007) and we use the fact that K is a bounded and \mathcal{F}^W -measurable and adapted process to obtain that

$$\begin{aligned} & |E((\partial_x^{n+2} - \partial_x^{n+1})BS(s, X_s, M_s^T, v_s)|\mathcal{G}_t)| \\ & \leq C \left((1 - \rho^2) \int_t^s \sigma_\theta^2 d\theta + \int_s^T a_\theta^2 d\theta \right)^{-\frac{1}{2}(n+1)}. \end{aligned}$$

We know that for all $\rho \in (-1, 1)$, $a_s^2 \geq C(\rho) \sigma_s^2$ for some positive constant $C(\rho)$. Then $\int_s^T a_\theta^2 d\theta \geq C(\rho) \int_s^T \sigma_\theta^2 d\theta$, from where the result follows. ■

Now we are able to prove the main result of this section, the extended Hull and White formula. We will need the following hypothesis:

(H1) The process $a^2 \in \mathbb{L}^{1,2}$.

Theorem 2 Consider the model (1) and assume that hypothesis (H1) holds. Then it follows that

$$\begin{aligned} V_t &= E(BS(t, X_t, M_t^T, v_t)|\mathcal{F}_t) \\ &+ \frac{1}{2} E \left\{ \rho \int_t^T e^{-r(s-t)} (\partial_{xxx}^3 - \partial_{xx}^2) BS(s, X_s, M_s^T, v_s) \sigma_s \Lambda_s^W ds \right. \\ &\left. + \int_t^T e^{-r(s-t)} \partial_K (\partial_{xx}^2 - \partial_x) BS(s, X_s, M_s^T, v_s) \Lambda_s^W m(T, s) ds \middle| \mathcal{F}_t \right\}, \quad (3) \end{aligned}$$

where $\Lambda_s^W := \left[D_s^W \int_s^T a^2(r) dr \right]$.

Proof: This proof is similar to the one of the main theorem in Alòs, León and Vives (2007), so we only sketch it. Notice that $BS(T, X_T, M_T^T, v_T) = V_T$. Then, from (2), we have

$$e^{-rt} V_t = E(e^{-rT} BS(T, X_T, M_T^T, v_T)|\mathcal{F}_t).$$

Now, using the Itô's formula to the process

$$t \rightarrow e^{-rt} BS(t, X_t, M_t^T, v_t)$$

and proceeding as in Alòs, León and Vives (2007) (see also Alòs and Nualart (1998), Alòs (2006) or Nualart (2006)), we can write

$$\begin{aligned}
& e^{-rT} BS(T, X_T, M_T^T, v_T) \\
&= e^{-rt} BS(t, X_t, M_t^T, v_t) \\
&+ \int_t^T e^{-rs} \mathcal{L}_{BS}(v_s^2) BS(s, X_s, M_s^T, v_s) ds \\
&+ \int_t^T e^{-rs} \partial_x BS(s, X_s, M_s^T, v_s) \sigma_s (\rho dW_s + \sqrt{1 - \rho^2} dB_s) \\
&+ \int_t^T e^{-rs} \partial_K BS(s, X_s, M_s^T, v_s) dM_s^T \\
&+ \int_t^T e^{-rs} \partial_{xK}^2 BS(s, X_s, M_s^T, v_s) d\langle M^T, X \rangle_s \\
&+ \frac{1}{2} \int_t^T e^{-rs} \partial_\sigma BS(s, X_s, M_s^T, v_s) \frac{v_s^2 - a_s^2}{v_s(T-s)} ds \\
&+ \int_t^T e^{-rs} \partial_{x\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\sigma_s \rho \Lambda_s^W}{2v_s(T-s)} ds \\
&+ \int_t^T e^{-rs} \partial_{K\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\Lambda_s^W m(T, s)}{2v_s(T-s)} ds \\
&+ \frac{1}{2} \int_t^T e^{-rs} (\partial_{xx}^2 - \partial_x) BS(s, X_s, M_s^T, v_s) (\sigma_s^2 - v_s^2) ds \\
&+ \frac{1}{2} \int_t^T e^{-rs} \partial_{KK}^2 BS(s, X_s, M_s^T, v_s) d\langle M^T, M^T \rangle_s.
\end{aligned}$$

Hence, the fact that $\mathcal{L}_{BS}(v_s^2)BS(s, X_s, M_s^T, v_s) = 0$, multiplying by e^{rt} and taking conditional expectations we can establish

$$\begin{aligned}
& E \left(e^{-r(T-t)} BS(T, X_T, M_T^T, v_T) \middle| \mathcal{F}_t \right) \\
&= E \left\{ BS(t, X_t, M_t^T, v_t) + \int_t^T e^{-r(s-t)} \partial_{xK}^2 BS(s, X_s, M_s^T, v_s) d\langle M^T, X \rangle_s \right. \\
&\quad + \frac{1}{2} \int_t^T e^{-r(s-t)} \partial_\sigma BS(s, X_s, M_s^T, v_s) \frac{v_s^2 - a_s^2}{v_s(T-s)} ds \\
&\quad + \int_t^T e^{-r(s-t)} \partial_{x\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\sigma_s \rho \Lambda_s^W}{2v_s(T-s)} ds \\
&\quad + \int_t^T e^{-r(s-t)} \partial_{K\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\Lambda_s^W m(T, s)}{2v_s(T-s)} ds \\
&\quad + \frac{1}{2} \int_t^T e^{-r(s-t)} (\partial_{xx}^2 - \partial_x) BS(s, X_s, M_s^T, v_s) (\sigma_s^2 - v_s^2) ds \\
&\quad \left. + \frac{1}{2} \int_t^T e^{-r(s-t)} \partial_{KK}^2 BS(s, X_s, M_s^T, v_s) d\langle M^T, M^T \rangle_s \middle| \mathcal{F}_t \right\}.
\end{aligned}$$

Consequently, the classical relationships between the greeks:

$$\begin{aligned}
\partial_{xx}^2 BS - \partial_x BS &= \partial_\sigma BS \frac{1}{\sigma(T-t)} \\
\partial_{xK}^2 BS &= -\partial_\sigma BS \frac{1}{K\sigma(T-t)} \\
\partial_{KK}^2 BS &= \partial_\sigma BS \frac{1}{K^2\sigma(T-t)}
\end{aligned}$$

give

$$\begin{aligned}
& E \left(e^{-r(T-t)} BS(T, X_T, M_T^T, v_T) \middle| \mathcal{F}_t \right) \\
&= E \left\{ BS(t, X_t, M_t^T, v_t) - \int_t^T e^{-r(s-t)} \partial_\sigma BS(s, X_s, M_s^T, v_s) \frac{1}{M_s^T v_s (T-s)} d\langle M^T, X \rangle_s \right. \\
&\quad + \frac{1}{2} \int_t^T e^{-r(s-t)} \partial_\sigma BS(s, X_s, M_s^T, v_s) \frac{v_s^2 - a_s^2}{v_s(T-s)} ds \\
&\quad + \int_t^T e^{-r(s-t)} \partial_{x\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\sigma_s \rho \Lambda_s^W}{2v_s(T-s)} ds \\
&\quad + \int_t^T e^{-r(s-t)} \partial_{K\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\Lambda_s^W m(T, s)}{2v_s(T-s)} ds \\
&\quad + \frac{1}{2} \int_t^T e^{-r(s-t)} \partial_\sigma BS(s, X_s, M_s^T, v_s) (\sigma_s^2 - v_s^2) \frac{1}{v_s(T-t)} ds \\
&\quad \left. + \frac{1}{2} \int_t^T e^{-r(s-t)} \partial_\sigma BS(s, X_s, M_s^T, v_s) \frac{1}{M_s^2 v_s (T-s)} d\langle M^T, M^T \rangle_s \middle| \mathcal{F}_t \right\}.
\end{aligned}$$

That is,

$$\begin{aligned}
& E \left(e^{-r(T-t)} BS(T, X_T, M_T^T, v_T) \middle| \mathcal{F}_t \right) \\
&= E \left\{ BS(t, X_t, M_t^T, v_t) + \int_t^T e^{-r(s-t)} \frac{\partial_\sigma BS(s, X_s, M_s^T, v_s)}{v_s(T-t)} \right. \\
&\quad \times \left[-\frac{d\langle M^T, X \rangle_s}{M_s^T} + \frac{1}{2} (v_s^2 - a_s^2) ds + \frac{1}{2} (\sigma_s^2 - v_s^2) ds + \frac{1}{2} \frac{d\langle M^T, M^T \rangle_s}{(M_s^T)^2} \right] \\
&\quad + \int_t^T e^{-r(s-t)} \partial_{x\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\sigma_s \rho \Lambda_s^W}{2v_s(T-s)} ds \\
&\quad \left. + \int_t^T e^{-r(s-t)} \partial_{K\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\Lambda_s^W m(T, s)}{2v_s(T-s)} ds \middle| \mathcal{F}_t \right\}.
\end{aligned}$$

Since, $a_s^2 ds := \sigma_s^2 ds - 2 \frac{d\langle M^T, X \rangle_s}{M_s^T} + \frac{d\langle M^T, M^T \rangle_s}{(M_s^T)^2}$ we obtain

$$\begin{aligned}
& E \left(e^{-r(T-t)} BS(T, X_T, M_T^T, v_T) \middle| \mathcal{F}_t \right) \\
&= E \left\{ BS(t, X_t, M_t^T, v_t) + \int_t^T e^{-r(s-t)} \partial_{x\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\sigma_s \rho \Lambda_s^W}{2v_s(T-s)} ds \right. \\
&\quad \left. + \int_t^T e^{-r(s-t)} \partial_{K\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\Lambda_s^W m(T, s)}{2v_s(T-s)} ds \middle| \mathcal{F}_t \right\},
\end{aligned}$$

as we wanted to prove. ■

Example 3 Assume the model (1) with constant volatility σ . We consider a call spread option with strike equal to $K_T = S'_T + K$, where K is a non-negative deterministic constant and S' is another stock price of the form $S'_t = \exp(X'_t)$, where

$$dX'_t = \left(r - \frac{(\sigma')^2}{2} \right) dt + \sigma' dW_t, \quad t \in [0, T],$$

for some positive constant σ' . Then we can easily check that $m(T, \theta) = \exp(r(T-\theta))S'_\theta \sigma'$, $M_\theta^T = \exp(r(T-\theta))S'_\theta + K$, and then

$$a_\theta^2 := \sigma^2 - \frac{2\rho\sigma\sigma' \exp(r(T-\theta))S'_\theta}{\exp(r(T-\theta))S'_\theta + K} + \frac{(\sigma')^2 (\exp(r(T-\theta))S'_\theta)^2}{(\exp(r(T-\theta))S'_\theta + K)^2}.$$

Notice that, if $K = 0$

$$a_\theta^2 := \sigma^2 - 2\rho\sigma\sigma' + (\sigma')^2$$

and $D_s^W a^2(\theta) = 0$. Then, the equality (3) reduces to

$$V_t = BS \left(t, X_t, \exp(r(T-t))S'_t, \sqrt{\sigma^2 - 2\rho\sigma\sigma' + (\sigma')^2} \right),$$

from where we recover the well-known Margrabe formula (see Margrabe (1978)).

Remark 4 Notice that, in the context of the previous example, when K is negative, the call option on the spread $S_T - S'_T$ is equivalent to the corresponding put option on the spread $S'_T - S_T$ with positive strike $-K$. Then, without loss of generality, we can assume that the spread option is written with a positive K .

4 Derivative of the implied volatility

Let $I_t(X_t)$ denote the implied volatility process, which satisfies by definition $V_t = BS(t, X_t, M_t, I_t(X_t))$. In this section we prove a formula for its at-the-money derivative that we use in Section 5 to study the short-time behavior of the implied volatility.

Proposition 5 Assume that the model (1) holds with $a \in \mathbb{L}^{1,2}$ and that, for every fixed $t \in [0, T)$, $\left(\int_t^T \sigma_\theta^2 d\theta\right)^{-1} < \infty$. Then it follows that

$$\frac{\partial I_t}{\partial X_t}(x_t^*) = \frac{E\left(\int_t^T e^{-r(s-t)} (\partial_x F(s, X_s, M_s^T, v_s) - \frac{1}{2} F(s, X_s, M_s^T, v_s)) ds \middle| \mathcal{F}_t\right)}{\partial_\sigma BS(t, x_t^*, M_t^T, I_t(x_t^*))} \Bigg|_{X_t=x_t^*}, \quad a.s.$$

where

$$F(s, X_s, M_s^T, v_s) = \frac{1}{2} [(\partial_{xxx}^3 - \partial_{xx}^2) BS(s, X_s, M_s^T, v_s) \sigma_s \rho \Lambda_s^W + \partial_K (\partial_{xx}^2 - \partial_x) BS(s, X_s, M_s^T, v_s) \Lambda_s^W m(T, s)]$$

and $x_t^* = \ln(M_t^T) - r(T - t)$.

Proof: Using Theorem 2 and the expression $V_t = BS(t, X_t, M_t^T, I_t(X_t))$ we obtain

$$\frac{\partial V_t}{\partial X_t} = \partial_x BS(t, X_t, M_t^T, I_t(X_t)) + \partial_\sigma BS(t, X_t, M_t^T, I_t(X_t)) \frac{\partial I_t}{\partial X_t}(X_t). \quad (4)$$

and

$$V_t = E(BS(t, X_t, M_t^T, v_t) | \mathcal{F}_t) + E\left(\int_t^T e^{-r(s-t)} F(s, X_s, M_s^T, v_s) ds \middle| \mathcal{F}_t\right),$$

which implies that

$$\frac{\partial V_t}{\partial X_t} = E(\partial_x BS(t, X_t, M_t^T, v_t) | \mathcal{F}_t) + E\left(\int_t^T e^{-r(s-t)} \partial_x F(s, X_s, M_s^T, v_s) ds \middle| \mathcal{F}_t\right). \quad (5)$$

We can check that the conditional expectation $E\left(\int_t^T e^{-r(s-t)} \partial_x F(s, X_s, M_s^T, v_s) ds \middle| \mathcal{F}_t\right)$ is well defined and finite a.s. due to the fact that $\left(\int_t^T \sigma_\theta^2 d\theta\right)^{-1} < \infty$. Thus, (4)

and (5) imply

$$\begin{aligned}
& \frac{\partial I_t}{\partial X_t}(x_t^*) \\
&= \frac{1}{\partial_\sigma BS(t, x_t^*, M_t^T, I_t(x_t^*))} \left[E(\partial_x BS(t, x_t^*, M_t^T, v_t) | \mathcal{F}_t) - \partial_x BS(t, x_t^*, M_t^T, I_t(x_t^*)) \right. \\
&\quad \left. + E\left(\int_t^T e^{-r(s-t)} \partial_x F(s, X_s, M_s^T, v_s) ds | \mathcal{F}_t \right) \right] \Bigg|_{X_t=x_t^*}. \tag{6}
\end{aligned}$$

Notice that

$$\begin{aligned}
& E(\partial_x BS(t, x_t^*, M_t^T, v_t) | \mathcal{F}_t) \\
&= \partial_x E(BS(t, x, M_t^T, v_t) | \mathcal{F}_t) \Big|_{x=x_t^*} = \partial_x BS(t, x, M_t^T, I_t^0(x)) \Big|_{x=x_t^*}, \tag{7}
\end{aligned}$$

where, by the Hull and White formula, $I_t^0(X_t)$ is the implied volatility of call option with constant strike M_t^T , for a certain stochastic volatility model where $\rho = 0$ and the volatility process is given by a_t . Thus,

$$\begin{aligned}
& \partial_x (BS(t, x, M_t^T, I_t^0(x)) \Big|_{x=x_t^*}) \\
&= \partial_x BS(t, x_t^*, M_t^T, I_t^0(x_t^*)) + \partial_\sigma BS(t, x_t^*, M_t^T, I_t^0(x_t^*)) \frac{\partial I_t^0}{\partial x}(x_t^*). \tag{8}
\end{aligned}$$

From Renault and Touzi (1996) we know that $\frac{\partial I_t^0}{\partial x}(x_t^*) = 0$. Then, (6), (7) and (8) imply that

$$\begin{aligned}
& \frac{\partial I_t}{\partial X_t}(x_t^*) \\
&= \frac{1}{\partial_\sigma BS(t, x_t^*, M_t^T, I_t(x_t^*))} \left[\partial_x BS(t, x_t^*, M_t^T, I_t^0(x_t^*)) - \partial_x BS(t, x_t^*, M_t^T, I_t(x_t^*)) \right. \\
&\quad \left. + E\left(\int_t^T e^{-r(s-t)} \partial_x F(s, X_s, M_s^T, v_s) ds | \mathcal{F}_t \right) \right] \Bigg|_{X_t=x_t^*}. \tag{9}
\end{aligned}$$

On the other hand, straightforward calculations lead us to

$$\partial_x BS(t, x_t^*, M_t^T, \sigma) = e^{x_t^*} N\left(\frac{1}{2}\sigma\sqrt{T-t}\right)$$

and

$$BS(t, x_t^*, M_t^T, \sigma) = e^{x_t^*} \left(N\left(\frac{1}{2}\sigma\sqrt{T-t}\right) - N\left(-\frac{1}{2}\sigma\sqrt{T-t}\right) \right).$$

Then

$$\partial_x BS(t, x_t^*, M_t^T, \sigma) = \frac{1}{2}(e^{x_t^*} + BS(t, x_t^*, M_t^T, \sigma)),$$

which yields

$$\begin{aligned}
& \partial_x BS(t, x_t^*, M_t^T, I_t^0(x_t^*)) - \partial_x BS(t, x_t^*, M_t^T, I_t(x_t^*)) \\
&= \frac{1}{2} (BS(t, x_t^*, M_t^T, I_t^0(x_t^*)) - BS(t, x_t^*, M_t^T, I_t(x_t^*))) \\
&= \frac{1}{2} E(BS(t, x_t^*, M_t^T, v_t) - V_t | \mathcal{F}_t) \\
&= -\frac{1}{2} E \left(\int_t^T e^{-r(s-t)} F(s, X_s, M_s^T, v_s) ds | \mathcal{F}_t \right) \Bigg|_{X_t=x_t^*}.
\end{aligned}$$

This, together with (9), implies that the result holds. ■

5 Short-time behaviour

The purpose of this section is to study the limit of $\frac{\partial I_t}{\partial X_t}(x_t^*)$ as $T \downarrow t$. The following result is part of the tool needed for our results.

Lemma 6 *Assume the model (1) is satisfied. Then $I_t(x_t^*)\sqrt{T-t} \rightarrow 0$ a.s. as $T \rightarrow t$.*

Proof: Notice that the fact that K is a square-integrable and continuous random process and the dominated convergence theorem lead to get

$$\begin{aligned}
V_t |_{X_t=x_t^*} &= E(e^{-r(T-t)}(e^{X_T} - K_T)_+ | \mathcal{F}_t) \Big|_{X_t=x_t^*} \\
&= E(e^{-r(T-t)}(e^{X_T - X_t} e^{-r(T-t)} M_t^T - K_T)_+ | \mathcal{F}_t) \Big|_{X_t=x_t^*} \\
&\leq E((e^{X_T - X_t} M_t^T - K_T e^{r(T-t)})_+ | \mathcal{F}_t) \Big|_{X_t=x_t^*} \\
&= E\left(\left((e^{X_T - X_t} - e^{r(T-t)})M_t^T + e^{r(T-t)}(M_t^T - K_T)\right)_+ | \mathcal{F}_t\right) \Big|_{X_t=x_t^*} \\
&\leq E(|e^{X_T - X_t} - e^{r(T-t)}| M_t^T | \mathcal{F}_t) \Big|_{X_t=x_t^*} \\
&\quad + E(|M_t^T - K_T| e^{r(T-t)} | \mathcal{F}_t) \Big|_{X_t=x_t^*} \\
&\leq M_t^T E(|e^{X_T - X_t} - e^{r(T-t)}| | \mathcal{F}_t) \Big|_{X_t=x_t^*} \\
&\quad + E(|M_t^T - K_T| e^{r(T-t)} | \mathcal{F}_t) \Big|_{X_t=x_t^*} \rightarrow 0 \text{ a.s.},
\end{aligned}$$

as $T \rightarrow t$. Hence, taking into account that, in the at-the-money case, $V_t |_{X_t=x_t^*} = BS(t, x_t^*, M_t^T, I_t(x_t^*))$, we deduce that

$$BS(t, x_t^*, M_t^T, I_t(x_t^*)) = 2M_t^T e^{-r(T-t)} \left[N\left(\frac{I(x_t^*)\sqrt{T-t}}{2}\right) - \frac{1}{2} \right] \rightarrow 0 \text{ a.s.},$$

and this allows us to complete the proof. ■

Henceforth we consider the following hypotheses:

(H1') $a^2 \in L^{2,2}$ and, moreover, there exists a positive constant C such that , for all $0 < s < \theta < r < T$,

$$|D_s^W a_r| + |D_\theta^W D_s^W a_r| \leq C.$$

Notice that this hypotheses implies that (H1) holds.

(H2) There exist two positive constants c_1, c_2 such that for all $r \in [0, T]$ $c_1 \leq \sigma_r \leq c_2$. Notice that, as for all $\rho \in (-1, 1)$, $a_s^2 \geq C(\rho) \sigma_s^2$ for some positive constant $C(\rho)$, this hypothesis implies that a_s^2 is lower bounded.

(H3) The process $m(T, \cdot) \in L^{1,2}$ and moreover, there exists a positive \mathcal{F}_t -adapted process C_t such that for all $T > s > r > t$,

$$E \left(|m(T, r)|^2 \middle| \mathcal{F}_t \right) + E \left(|D_s^W m(T, r)|^2 \middle| \mathcal{F}_t \right) \leq C_t.$$

Proposition 7 *Assume that the model (1) and Hypotheses (H1')-(H3) hold. Also assume that there is a constant $c > 0$ such that $c < K_t$, for all $t \in [0, T]$. Then*

$$\begin{aligned} & \partial_\sigma BS(t, x_t^*, M_t^T, I_t(x_t^*)) \frac{\partial I_t}{\partial X_t}(x_t^*) \\ &= \frac{1}{2} E \left(\rho \left(\partial_x - \frac{1}{2} \right) (\partial_{xxx}^3 - \partial_{xx}^2) BS(t, x_t^*, M_t^T, v_t) \int_t^T \sigma_s \Lambda_s^W ds \right. \\ & \quad \left. + \partial_K \left(\partial_x - \frac{1}{2} \right) (\partial_{xx}^2 - \partial_x) BS(t, x_t^*, M_t^T, v_t) \int_t^T \Lambda_s^W m(T, s) ds \middle| \mathcal{F}_t \right) \\ & \quad + O(T - t). \end{aligned}$$

as $T \rightarrow t$.

Proof: Proposition 5 gives us that

$$\begin{aligned} & \partial_\sigma BS(t, x_t^*, M_t^T, I_t(x_t^*)) \frac{\partial I_t}{\partial X_t}(x_t^*) \\ &= \frac{1}{2} E \left(\rho \int_t^T e^{-r(s-t)} (\partial_x - \frac{1}{2}) (\partial_{xxx}^3 - \partial_{xx}^2) BS(s, X_s, M_s^T, v_s) \sigma_s \Lambda_s^W ds \right. \\ & \quad \left. + \int_t^T e^{-r(s-t)} (\partial_x - \frac{1}{2}) \partial_K (\partial_{xx}^2 - \partial_x) BS(s, X_s, M_s^T, v_s) \right. \\ & \quad \left. \times \Lambda_s^W m(T, s) ds \middle| \mathcal{F}_t \right) \Big|_{X_t=x_t^*} =: T_1 + T_2. \end{aligned}$$

Now the proof is decomposed into two steps.

Step 1. Here we see that

$$T_1 = \frac{\rho}{2} E \left(L(t, x_t^*, M_t^T, v_t) \int_t^T \sigma_s \Lambda_s^W ds | \mathcal{F}_t \right) + O(T-t), \quad (10)$$

where $L(s, X_s, M_s^T, v_s) = (\partial_x - \frac{1}{2}) (\partial_{xxx}^3 - \partial_{xx}^2) BS(s, X_s, M_s^T, v_s)$. In fact, applying Itô formula to

$$\rho e^{-rs} L(s, X_s, M_s^T, v_s) \left(\int_s^T \sigma_r \Lambda_r^W dr \right)$$

as in the proof of Theorem 2 and taking conditional expectations with respect to \mathcal{F}_t , we obtain that

$$\begin{aligned} & \frac{\rho}{2} E \left(\int_t^T e^{-r(s-t)} L(s, X_s, M_s^T, v_s) \sigma_s \Lambda_s^W ds | \mathcal{F}_t \right) \\ &= \frac{\rho}{2} E \left(L(t, X_t, M_t^T, v_t) \left(\int_t^T \sigma_s \Lambda_s^W ds \right) | \mathcal{F}_t \right) \\ & \quad + \frac{\rho^2}{4} E \left(\int_t^T e^{-r(s-t)} (\partial_{xxx}^3 - \partial_{xx}^2) L(s, X_s, M_s^T, v_s) \sigma_s \Lambda_s^W \right. \\ & \quad \quad \times \left. \left(\int_s^T \sigma_r \Lambda_r^W dr \right) ds | \mathcal{F}_t \right) \\ & \quad + \frac{\rho}{4} E \left(\int_t^T e^{-r(s-t)} \partial_K (\partial_{xx}^2 - \partial_x) L(s, X_s, M_s^T, v_s) \Lambda_s^W m(T, s) \right. \\ & \quad \quad \times \left. \left(\int_s^T \Lambda_r^W \sigma_r dr \right) ds | \mathcal{F}_t \right) \\ & \quad + \frac{\rho^2}{2} E \left(\int_t^T e^{-r(s-t)} \partial_x L(s, X_s, M_s^T, v_s) \sigma_s \right. \\ & \quad \quad \times \left. \left(\int_s^T (D_s^W \Lambda_r^W) \sigma_r dr \right) ds | \mathcal{F}_t \right) \\ & \quad + \frac{\rho}{2} E \left(\int_t^T e^{-r(s-t)} \partial_K L(s, X_s, M_s^T, v_s) m(T, s) \right. \\ & \quad \quad \times \left. \left(\int_s^T (D_s^W \Lambda_r^W) \sigma_r dr \right) ds | \mathcal{F}_t \right) \\ &= \frac{1}{2} E \left(L(t, X_t, M_t^T, v_t) \left(\int_t^T \rho \sigma_s \Lambda_s^W ds \right) | \mathcal{F}_t \right) \\ & \quad + S_1 + S_2 + S_3 + S_4. \end{aligned}$$

Using the proof of Lemma 1 and Hypotheses (H1') and (H2), we can write

$$\begin{aligned}
|S_1| &= \left| \frac{\rho^2}{4} E \left(\int_t^T e^{-r(s-t)} E \left[(\partial_{xxx}^3 - \partial_{xx}^2) L(s, X_s, M_s^T, v_s) \middle| \mathcal{G}_t \right] \right. \right. \\
&\quad \left. \left. \times \left(\int_s^T \Lambda_r^W \sigma_r dr \right) \Lambda_s^W \sigma_s ds \middle| \mathcal{F}_t \right) \right| \\
&\leq C \sum_{k=4}^6 E \left[\int_t^T \left(\int_s^T a_\theta^2 d\theta \right)^{-\frac{k}{2}} \left| \left(\int_s^T \Lambda_r^W \sigma_r dr \right) \Lambda_s^W \sigma_s ds \middle| \mathcal{F}_t \right| \right] \\
&\leq C \sum_{k=4}^6 E \left[\int_t^T (T-s)^{-\frac{k}{2}} \left| \left(\int_s^T \Lambda_r^W \sigma_r dr \right) \Lambda_s^W \sigma_s ds \middle| \mathcal{F}_t \right| \right]
\end{aligned}$$

Hence, using Hypotheses (H1'), (H2), and (H3), we can write

$$|S_1| \leq C \sum_{k=4}^6 (T-t)^{-\frac{k}{2}+4} = O(T-t).$$

Similarly, we have

$$\begin{aligned}
|S_2| &= \left| \frac{\rho}{4} E \left(\int_t^T e^{-r(s-t)} E \left[\partial_K (\partial_{xx}^2 - \partial_x) L(s, X_s, M_s^T, v_s) \middle| \mathcal{G}_t \right] \right. \right. \\
&\quad \left. \left. \times \left(\int_s^T \Lambda_r^W \sigma_r dr \right) \Lambda_s^W m(T, s) ds \middle| \mathcal{F}_t \right) \right|.
\end{aligned}$$

Therefore, the relation

$$\frac{\partial^2 BS(t, x, K, \sigma)}{\partial_x \partial_K} = \frac{1}{k} \left(\frac{\partial BS(t, x, K, \sigma)}{\partial_x} - \frac{\partial^2 BS(t, x, K, \sigma)}{\partial_x^2} \right),$$

togheter with the hypotheses of the Proposition, implies

$$|S_2| \leq C_t \sum_{k=3}^6 E \left(\int_t^T (T-s)^{-\frac{k}{2}+3} |m(T, s)| \middle| \mathcal{F}_t \right) = O(T-t).$$

In a similar way,

$$\begin{aligned}
|S_3| &= \frac{\rho^2}{2} E \left(\int_t^T e^{-r(s-t)} \partial_x L(s, X_s, M_s, v_s) \sigma_s \right. \\
&\quad \left. \times \left(\int_s^T (D_s^W \Lambda_r^W) \sigma_r dr \right) ds \middle| \mathcal{F}_t \right) \Big|_{X_t=x_t^*} \\
&\leq C \sum_{k=3}^4 \left| E \left(\int_t^T (T-s)^{-\frac{k}{2}} \sigma_s \left(\int_s^T (D_s^W \Lambda_r^W) \sigma_r dr \right) ds \middle| \mathcal{F}_t \right) \right| \\
&\leq C \sum_{k=3}^4 \left(\int_t^T (T-s)^{-\frac{k}{2}+2} \right) = O(T-t).
\end{aligned}$$

Finally, the same arguments give us that

$$|S_4| = O(T - t).$$

Step 2. In order to finish the proof we only need to proceed as in Step 1. Here we see that

$$T_2 = \frac{1}{2}E \left(P(t, x_t^*, M_t^T, v_t) \int_t^T \Lambda_s^W m(T, s) ds | \mathcal{F}_t \right) + O(T - t), \quad (11)$$

where $P(s, X_s, M_s^T, v_s) = (\partial_x - \frac{1}{2})\partial_K (\partial_{xx}^2 - \partial_x) BS(s, X_s, M_s^T, v_s)$. In fact, applying Itô formula to

$$e^{-rs} P(s, X_s, M_s^T, v_s) \left(\int_s^T m(T, r) \Lambda_r^W dr \right)$$

as in the proof of Theorem 2 and taking conditional expectations with respect to \mathcal{F}_t , we obtain that

$$\begin{aligned} & \frac{1}{2}E \left(\int_t^T e^{-r(s-t)} P(s, X_s, M_s^T, v_s) \Lambda_s^W m(T, s) ds \Big| \mathcal{F}_t \right) \\ &= \frac{1}{2}E \left(P(t, X_t, M_t^T, v_t) \left(\int_t^T m(T, s) \Lambda_s^W ds \right) \Big| \mathcal{F}_t \right) \\ & \quad + \frac{\rho}{4}E \left(\int_t^T e^{-r(s-t)} (\partial_{xxx}^3 - \partial_{xx}^2) P(s, X_s, M_s^T, v_s) \sigma_s \Lambda_s^W \right. \\ & \quad \quad \times \left. \left(\int_s^T m(T, r) \Lambda_r^W dr \right) ds \Big| \mathcal{F}_t \right) \\ & \quad + \frac{1}{4}E \left(\int_t^T e^{-r(s-t)} \partial_K (\partial_{xx}^2 - \partial_x) P(s, X_s, M_s^T, v_s) \Lambda_s^W m(T, s) \right. \\ & \quad \quad \times \left. \left(\int_s^T \Lambda_r^W m(T, r) dr \right) ds \Big| \mathcal{F}_t \right) \\ & \quad + \frac{\rho}{2}E \left(\int_t^T e^{-r(s-t)} \partial_x P(s, X_s, M_s^T, v_s) \sigma_s \right. \\ & \quad \quad \times \left. \left(\int_s^T D_s^W (\Lambda_r^W m(T, r)) dr \right) ds \Big| \mathcal{F}_t \right) \\ & \quad + \frac{1}{2}E \left(\int_t^T e^{-r(s-t)} \partial_K P(s, X_s, M_s^T, v_s) m(T, s) \right. \\ & \quad \quad \times \left. \left(\int_s^T D_s^W (\Lambda_r^W m(T, r)) dr \right) ds \Big| \mathcal{F}_t \right). \end{aligned}$$

Now, following the same arguments as in Step 1 the proof is complete. ■

Remark 8 *This proof only needs some integrability and regularity conditions. So, depending on the coefficients of the model (1) and the process K , Hypotheses (H1')-(H3) can be substituted by appropriate integrability conditions.*

Now we can state the main result of this paper. Toward this end, we need to state the following assumptions:

(H4) Assume that $m(\cdot, \cdot)$ has continuous paths and that, for each $t \in [0, T]$ fixed,

$$\sup_{t < \theta \wedge s \wedge r < T} E \left(\sigma_s a_r - \frac{\sigma_t}{\tilde{a}_t} a_\theta^2 \middle| \mathcal{F}_t \right) \rightarrow 0 \quad \text{as } T \rightarrow t, \text{ a.s.}$$

and

$$\sup_{t < \theta \wedge s \wedge r < T} E \left(m(T, s) a_r - \frac{m(t, t)}{\tilde{a}_t} a_\theta^2 \middle| \mathcal{F}_t \right) \rightarrow 0 \quad \text{as } T \rightarrow t, \text{ a.s.}$$

where, by convention,

$$\tilde{a}_t := \sigma_t^2 - 2\rho\sigma_t \frac{m(t, t)}{K_t} + \frac{m^2(t, t)}{K_t^2}$$

(H5) There exists a \mathcal{F}_t -measurable random variable $D_t^+ a_t$ such that, for every fixed $t > 0$,

$$\sup_{t < s < r < T} |E((D_s^W a_r - D_t^+ a_t) | \mathcal{F}_t)| \rightarrow 0, \quad \text{a.s.}$$

as $T \rightarrow t$.

Theorem 9 *Consider the model (1) and suppose that Hypotheses (H1')-(H5) hold and there exists a positive constant c such that $c < K$. Then*

$$\lim_{T \rightarrow t} \frac{\partial I_t}{\partial X_t}(x_t^*) = \frac{1}{2} \left(\frac{m(t, t)}{K_t} - \rho\sigma_t \right) \frac{D_t^+ a_t}{\tilde{a}_t^2}. \quad (12)$$

Proof: We can write

$$\partial_\sigma BS(t, x_t^*, M_t^T, I_t(x_t^*)) = \frac{M_t^T e^{-r(T-t)} e^{-\frac{I_t(x_t^*)^2(T-t)}{8}} \sqrt{T-t}}{\sqrt{2\pi}},$$

$$\left(\partial_x - \frac{1}{2} \right) (\partial_{xxx}^3 - \partial_{xx}^2) BS(t, x_t^*, M_t^T, v_t) = -M_t^T e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_t^2(T-t)}{8}} v_t^{-3} (T-t)^{-\frac{3}{2}}$$

and

$$\begin{aligned} & \partial_K \left(\partial_x - \frac{1}{2} \right) (\partial_{xx}^2 - \partial_x) BS(t, x_t^*, M_t^T, v_t) \\ &= M_t^T e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_t^2(T-t)}{8}} v_t^{-1} (T-t)^{-\frac{1}{2}} \left(\frac{1}{M_t^T v_t^2 (T-t)} \right). \end{aligned}$$

Then we can write, due to Lemma 6 and Proposition 7,

$$\begin{aligned}
& \frac{\partial I_t}{\partial X_t}(x_t^*) \\
&= -\frac{\rho}{2} e^{\frac{I_t(x_t^*)^2(T-t)}{8}} (T-t)^{-2} E\left(e^{-\frac{v_t^2(T-t)}{8}} v_t^{-3} \int_t^T \sigma_s \Lambda_s^W ds \middle| \mathcal{F}_t\right) \\
&\quad + \frac{1}{2} e^{\frac{I_t(x_t^*)^2(T-t)}{8}} (T-t)^{-1} E\left(e^{-\frac{v_t^2(T-t)}{8}} v_t^{-1} \left(\frac{1}{M_t^T v_t^2 (T-t)}\right) \int_t^T \Lambda_s^W m(T, s) ds \middle| \mathcal{F}_t\right) \\
&\quad + O(T-t)^{\frac{1}{2}} \\
&= : S_1 + S_2 + O(T-t)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 6, we know that $I_t(x_t^*)^2(T-t) \rightarrow 0$ a.s. as $T \rightarrow t$. Then,

$$\lim_{T \rightarrow t} S_1 = -\frac{\rho}{2} \lim_{T \rightarrow t} \left[(T-t)^{-2} E\left(e^{-\frac{v_t^2(T-t)}{8}} v_t^{-3} \int_t^T \sigma_s \Lambda_s^W ds \middle| \mathcal{F}_t\right) \right]$$

and

$$\begin{aligned}
\lim_{T \rightarrow t} S_2 &= \frac{1}{2} \lim_{T \rightarrow t} \left[(T-t)^{-1} E\left(e^{-\frac{v_t^2(T-t)}{8}} v_t^{-1} \left(\frac{1}{M_t^T v_t^2 (T-t)}\right) \right. \right. \\
&\quad \left. \left. \times \int_t^T \Lambda_s^W m(T, s) ds \middle| \mathcal{F}_t\right) \right]. \tag{13}
\end{aligned}$$

Now, let us see that

$$\lim_{T \rightarrow t} \left(S_1 + \frac{\rho \sigma_t}{2 \tilde{a}_t^2} D_t^+ a_t \right) = 0 \text{ a.s.} \tag{14}$$

In fact, we can establish

$$\lim_{T \rightarrow t} \left(S_1 + \frac{\rho \sigma_t}{2 \tilde{a}_t^2} D_t^+ a_t \right) = \lim_{T \rightarrow t} E \left(A_T B_T + \frac{\rho \sigma_t}{2 \tilde{a}_t^2} D_t^+ a_t \middle| \mathcal{F}_t \right)$$

where

$$A_T := \rho \exp\left(-\frac{v_t^2(T-t)}{8}\right) \frac{1}{v_t}$$

and

$$B_T := -\frac{1}{v_t^2(T-t)^2} \int_t^T \int_s^T a_r \sigma_s D_s^W a_r dr ds.$$

Consequently

$$\begin{aligned}
& \lim_{T \rightarrow t} E \left(A_T B_T + \frac{\rho \sigma_t}{2 \tilde{a}_t^2} D_t^+ a_t \middle| \mathcal{F}_t \right) \\
&= \lim_{T \rightarrow t} E \left(\left(A_T - \frac{\rho}{\tilde{a}_t} \right) B_T \middle| \mathcal{F}_t \right) + \frac{\rho}{\tilde{a}_t} \lim_{T \rightarrow t} E \left(B_T + \frac{\sigma_t}{2 \tilde{a}_t} D_t^+ a_t \middle| \mathcal{F}_t \right) \\
&= \lim_{T \rightarrow t} U_1 + \frac{\rho}{\tilde{a}_t} \lim_{T \rightarrow t} U_2.
\end{aligned}$$

Applying Schwartz inequality for conditional expectation, it follows that

$$U_1 \leq \left[E \left(\left(A_T - \frac{\rho}{\tilde{a}_t} \right)^2 \middle| \mathcal{F}_t \right) \right]^{\frac{1}{2}} [E (B_T^2 | \mathcal{F}_t)]^{\frac{1}{2}}.$$

From the dominated convergence theorem and (H2), it is easy to see that $E \left(\left(A_T - \frac{\rho}{\tilde{a}_t} \right)^2 \middle| \mathcal{F}_t \right)$ tends to zero a.s. as $T \rightarrow t$, and a simple calculation gives us that (H1') and (H2) imply that $E (B_T^2 | \mathcal{F}_t)$ is bounded, from where we deduce that $\lim_{T \rightarrow t} U_1 = 0$.

Observe that we also have,

$$\begin{aligned} |U_2| &= \left| \frac{1}{(T-t)^2} E \left(\int_t^T \int_s^T \left(\frac{\sigma_s a_r}{v_t^2} D_s^W a_r - \frac{\sigma_t}{\tilde{a}_t} D_t^+ a_t \right) dr ds \middle| \mathcal{F}_t \right) \right| \\ &\leq \frac{C}{(T-t)^2} \left| E \left(\int_t^T \int_s^T \left(\frac{\sigma_s a_r}{v_t^2} - \frac{\sigma_t}{\tilde{a}_t} \right) D_s^W a_r dr ds \middle| \mathcal{F}_t \right) \right| \\ &\quad + \frac{C}{(T-t)^2} \left| E \left(\int_t^T \int_s^T (D_s^W a_r - D_t^+ a_t) \middle| \mathcal{F}_t \right) dr ds \right| \\ &=: |U_{2,1}| + |U_{2,2}|. \end{aligned}$$

Using Hypotheses (H1') and (H2) we obtain that

$$\begin{aligned} |U_{2,1}| &\leq \frac{C}{(T-t)^2} E \left(\int_t^T \int_s^T \left| \frac{\sigma_s a_r}{v_t^2} - \frac{\sigma_t}{\tilde{a}_t} \right| dr ds \middle| \mathcal{F}_t \right) \\ &\leq \frac{C}{(T-t)^2} E \left(\int_t^T \int_s^T \left| \sigma_s a_r - \frac{\sigma_t}{\tilde{a}_t} v_t^2 \right| dr ds \middle| \mathcal{F}_t \right) \\ &= \frac{C}{(T-t)^2} E \left(\int_t^T \int_s^T \left| \sigma_s a_r - \frac{\sigma_t}{\tilde{a}_t (T-t)} \int_t^T a_\theta^2 d\theta \right| dr ds \middle| \mathcal{F}_t \right) \\ &\leq \frac{C}{(T-t)^3} \int_t^T \int_s^T \int_t^T E \left(\left| \sigma_s a_r - \frac{\sigma_t}{\tilde{a}_t} a_\theta^2 \right| \middle| \mathcal{F}_t \right) d\theta dr ds, \end{aligned}$$

which tends to zero, a.s. as $T \rightarrow t$, because of Hypothesis (H4). Similarly,

$$|U_{2,2}| \leq \frac{C}{(T-t)^2} \left| \int_t^T \int_s^T E \left((D_s^W a_r - D_t^+ a_t) \middle| \mathcal{F}_t \right) dr ds \right|,$$

which tends to zero by Hypothesis (H5). Thus we have proved (14) is true.

On the other hand, by (13) we can write

$$\lim_{T \rightarrow t} \left(S_2 - \frac{m(t, t)}{2K_t \tilde{a}_t^2} D^+ a_t \right) = \lim_{T \rightarrow t} E \left(A_T B_T - \frac{m(t, t)}{2K_t \tilde{a}_t^2} D^+ a_t \middle| \mathcal{F}_t \right)$$

but now

$$A_T := \exp\left(-\frac{v_t^2(T-t)}{8}\right) \frac{1}{M_t^T v_t}$$

and

$$B_T := \frac{1}{v_t^2(T-t)^2} \int_t^T \int_s^T a_r m(T, s) D_s^W a_r dr ds.$$

Finally, proceeding similarly as before, we have (13) yields that S_2 converges to $\frac{m(t,t)}{2K_t a_t^2} D^+ a_t$, which, together with (14), implies that (12) is satisfied. ■

6 Application to the study of spread options

This section is devoted to apply the previous results to study the implied volatility behaviour for spread options. This study allows us to predict when the Kirk's approximation formula for spread options may fail. Moreover, we will see how this analysis gives us a tool to improve Kirk's formula.

6.1 Short-time behaviour of the implied volatility for spread options

Consider an spread option with $K_T = S'_T + K$ as in Example 3. For the sake of simplicity we will assume the interest rate $r = 0$. Then it is easy to see that

$$a_t^2 := \sigma^2 - 2\rho\sigma\sigma' \frac{S'_t}{S'_t + K} + (\sigma')^2 \frac{(S'_t)^2}{(S'_t + K)^2}.$$

We can easily check that, for $\theta < t$

$$\begin{aligned} D_\theta^W a_t^2 &= \left(-2\rho\sigma\sigma' \frac{K}{(S'_t + K)^2} + 2(\sigma')^2 \left(\frac{S'_t}{S'_t + K} \right) \frac{K}{(S'_t + K)^2} \right) \sigma' S'_t \\ &= 2(\sigma')^2 \left(-\rho\sigma + \sigma' \left(\frac{S'_t}{S'_t + K} \right) \right) \frac{S'_t K}{(S'_t + K)^2}. \end{aligned}$$

Hence, we deduce that

$$\begin{aligned} D_\theta^W a_t &= D_\theta^W \sqrt{a_t^2} = \frac{D_r^W a_t^2}{2\sqrt{a_t^2}} \\ &= \frac{1}{\sqrt{a_t^2}} \left(-\rho\sigma + \sigma' \left(\frac{S'_t}{S'_t + K} \right) \right) (\sigma')^2 \frac{S'_t K}{(S'_t + K)^2}. \end{aligned}$$

Then, from Theorem 9, we get

$$\begin{aligned}
& \lim_{T \rightarrow t} \frac{\partial I_t}{\partial X_t}(x_t^*) \\
&= \frac{1}{2} \left(\frac{m(t, t)}{K_t} - \rho\sigma \right) \frac{D_t^+ a_t}{\tilde{a}_t^2} \\
&= \frac{1}{2} \left(\sigma' \left(\frac{S'_t}{S'_t + K} \right) - \rho\sigma \right)^2 \frac{1}{(\sqrt{a_t^2})^3} (\sigma')^2 \frac{S'_t K}{(S'_t + K)^2}. \quad (15)
\end{aligned}$$

Remark 10 Notice that the above quantity is always positive. In the following examples we will study its behaviour as a function of K and ρ .

Example 11 In Figure 1 we plot $\lim_{T \rightarrow t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ as a function of K for $\rho = 0.9$ (solid) and $\rho = 1$ (dash), and with $S_t = 100, \sigma = 0.5$ and $\sigma' = 0.4$. We can observe the limit skew $\lim_{T \rightarrow t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ is zero in the case $K = 0$. This was expected from Example 3, where we found that in this case the implied volatility is constant, and then $\frac{\partial I_t}{\partial X_t} = 0$. Notice also that, even this skew increases with K , this increment seems to be clearly bigger in the completely correlated case $\rho = 1$.

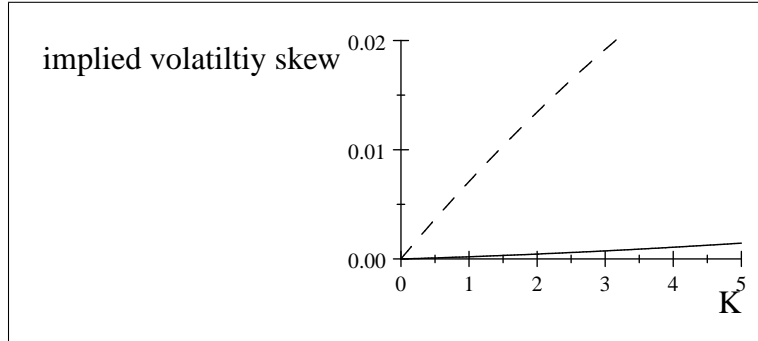


Figure 1: $\lim_{T \rightarrow t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ as a function of K for $\rho = 0.9$ (solid) and $\rho = 1$ (dash). Here $\sigma = 0.5, \sigma' = 0.4$.

Example 12 In Figure 2 we plot $\lim_{T \rightarrow t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ as a function of ρ for $K = 5$ (solid) and $K = 10$ (dash) and for the same parameter values as in Fig.1. We can observe the limit skew $\lim_{T \rightarrow t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ has its maximum in the completely correlated case $\rho = 1$. Notice that this means that the constant volatility approximation given by Kirk's formula is expected to be less accurate in this case. This fact is consistent numerical empirical evidence (see for example Baeva (2011) and Borovkova (2007)).

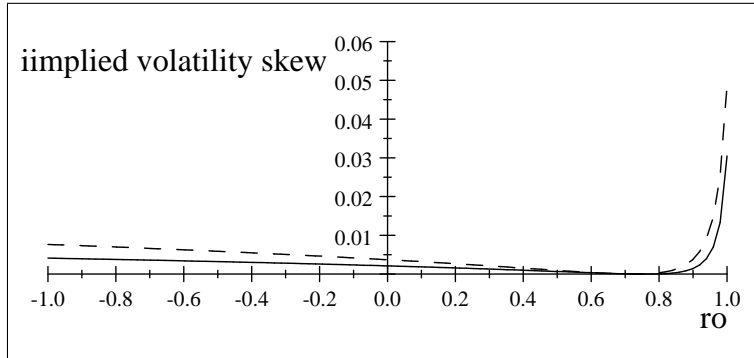


Figure 2: $\lim_{T \rightarrow t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ as a function of ρ for $K = 5$ (solid) and $K = 10$ (dash). Here $\sigma = 0.5, \sigma' = 0.4$.

Example 13 In Figure 3 we plot $\lim_{T \rightarrow t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ as a function of ρ and K for the same parameter values as in Fig.1 and Fig. 2. Notice that this limit skew is substantially bigger near the case $\rho = 1$.

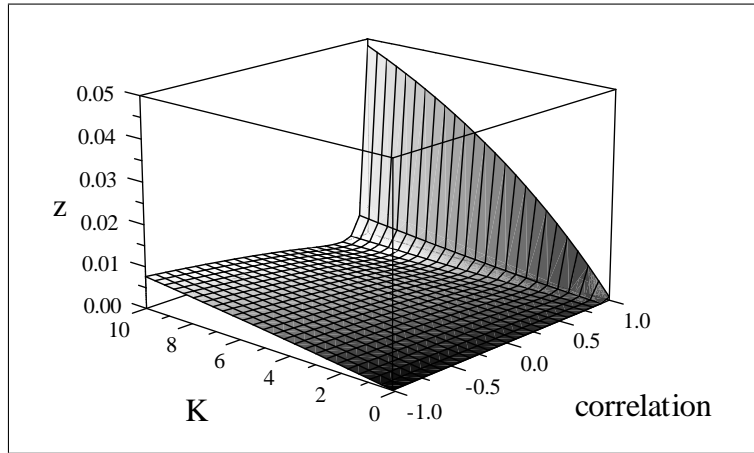


Figure 3: $\lim_{T \rightarrow t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ as a function of ρ and K .

6.2 Applications to the study of the accuracy of Kirk's formula

With the above notations, the Kirk approximation for an spread option can be written as:

$$BS(t, X_t, M_t^T, \sqrt{a_t^2}).$$

It is well-known that Kirk's formula is a very accurate approximation for spread options given its simplicity (see for example Baeva (2011), Bjerksund and Stensland (2011) or Carmona and Durrleman (2011)). Nevertheless, it is well-known it may fail for highly correlated assets (see for example Baeva (2011)). The results in the above sections give an analytical reason for this phenomenon.

In fact, notice that $\sqrt{a_t^2}$ (the volatility parameter in the Kirk's formula) is a process that does not depend on X_t nor on the time to maturity $T - t$. Then, Kirk's formula may not reproduce the short-time volatility skews that we have seen appear in the highly correlated case (ρ close to 1) and we can expect it can fail when ρ is near one. In the following examples, we compare Kirk's approximation prices with the ones obtained with a Monte-Carlo simulation procedure with 100,000 trials. We use these simulation results as the benchmark for the true spread option value

Example 14 *In the following table we can compare the prices given by Kirk's formula and by the Monte-Carlo simulations, for different values for K and ρ . Here $X_t = \ln(100)$, $S_t' = 100$, $r = 0$, $T - t = 0.5$, $\sigma = 0.5$ and $\sigma' = 0.4$. The difference is given in % of the Monte-Carlo price. As expected from our analytical study, the errors increase strongly when the correlation ρ is close to 1.*

K/ρ		0.60	0.98	0.99	0.999
5	Monte-Carlo	9,4564	2,1890	1,8386	1,5011
	Kirk	9,4176	2,2159	1,8775	1,5420
	Error	-0,410%	1,230%	2,117%	2,725%
10	Monte-Carlo	7,6404	1,2714	1,0207	0,7934
	Kirk	7,5988	1,3326	1,1015	0,8848
	Error	-0,545%	4,814%	7,913%	11,516%

Notice that our study of the volatility skew gives not only a theoretical understanding of it goodness-of-fit of Kirk's approximation, but also gives us some hints to improve it. In fact, from (15) and by using Taylor expansions we can expect that for small times to maturities and for near-the-money options, the expression

$$\hat{I}_t(X_t) := \sqrt{a_t^2} + \frac{1}{2} \left(\sigma' \left(\frac{S_t'}{S_t' + K} \right) - \rho\sigma \right)^2 \frac{1}{(\sqrt{a_t^2})^3} (\sigma')^2 \frac{S_t' K}{(S_t' + K)^2} (X_t - x_t^*)$$

can be a reasonable approximation for the implied volatility $I_t(X_t)$. In fact, let us consider the modified Kirk approximation given by

$$BS(t, X_t, M_t^T, \hat{I}_t(X_t)).$$

In the following example we will check numerically the goodness-of-fit of this approximation.

Example 15 *In the following table we can compare the prices given by the modified Kirk's formula and by the Monte-Carlo simulations, for different values for K and ρ and for the same parameters as in Example 14. Notice that we have reduced significantly the error of approximation with respect to the results in this last example, specially in the case of highly correlated assets.*

K/ρ		0.60	0.98	0.99	0.999
5	Monte-Carlo	9,4564	2,1890	1,8386	1,5011
	Modified Kirk	9,4255	2,2067	1,8309	1,4829
	Error	$-0,327\%$	$0,809\%$	$0,804\%$	$0,414\%$
10	Monte-Carlo	7,6404	1,2714	1,0207	0,7934
	Modified Kirk	7,6060	1,2888	1,0367	0,8210
	Error	$-0,451\%$	$1,368\%$	$1,660\%$	$1,400\%$

7 Conclusions

We have used the Malliavin calculus techniques to find an expression for the short-time implied volatility skew of options with random strikes. In particular, we have seen that this analytical study gives us a key to understand why some approximation formulas may fail for some sets of parameters. As an application, we have seen that this skew is very pronounced for spread options in the high correlation case, which explains why a constant volatility approximation as Kirk's formula cannot be accurate in this case. Finally, our approach gives us some hints to improve this estimate, by introducing a skew in the approximation of the corresponding implied volatility. Our preliminary numerical analysis shows that this can be a natural and efficient way to improve Kirk's formula. A precise development of this improvement is left for future research.

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