Chapter

Sets of Fractional Operators and Some of Their Applications

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Abstract

This chapter presents one way to define Abelian groups of fractional operators isomorphic to the group of integers under addition through a family of sets of fractional operators and a modified Hadamard product, as well as one way to define finite Abelian groups of fractional operators through sets of positive residual classes less than a prime number. Furthermore, it is presented one way to define sets of fractional operators which allow generalizing the Taylor series expansion of a vector-valued function in multi-index notation, as well as one way to define a family of fractional fixed-point methods and determine their order of convergence analytically through sets.

Keywords: fractional operators, set theory, group theory, fractional iterative methods, fractional calculus of sets

1. Introduction

In one dimension, a fractional derivative may be considered in a general way as a parametric operator of order α, such that it coincides with conventional derivatives when α is a positive integer n. So, when it is not necessary to explicitly specify the form of a fractional derivative, it is usually denoted as follows

$$\frac{d^\alpha}{dx^\alpha}. \tag{1}$$

On the other hand, a fractional differential equation is an equation that involves at least one differential operator of order α, with $(n-1) < \alpha \le n$ for some positive integer n, and it is said to be a differential equation of order α if this operator is the highest order in the equation. The fractional operators have many representations [1–3], but one of their fundamental properties is that they allow retrieving the results of conventional calculus when $\alpha \to n$. For example, let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L^1_{loc}(a, b)$, where $L^1_{loc}(a, b)$ denotes the space of locally integrable functions on the open interval $(a, b) \subset \Omega$. One of the fundamental operators of fractional calculus is the operator Riemann−Liouville fractional integral, which is defined as follows [4, 5]:

$${}_aI_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{2}$$

where Γ denotes the Gamma function. It is worth mentioning that the above operator is a fundamental piece to construct the operator Riemann−Liouville fractional derivative, which is defined as follows [4, 6]:

$$ {}_aD_x^\alpha f(x) := \begin{cases} {}_aI_x^{-\alpha}f(x), & \text{if} \quad \alpha < 0 \\ \dfrac{d^n}{dx^n}\left({}_aI_x^{n-\alpha}f(x)\right), & \text{if} \quad \alpha \geq 0 \end{cases}, \tag{3} $$

where $n = \lceil \alpha \rceil$ and ${}_aI_x^0 f(x) := f(x)$. On the other hand, let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function n-times differentiable such that $f, f^{(n)} \in L_{loc}^1(a, b)$. Then, the Riemann−Liouville fractional integral also allows constructing the operator Caputo fractional derivative, which is defined as follows [4, 6]:

$$ {}_a^CD_x^\alpha f(x) := \begin{cases} {}_aI_x^{-\alpha}f(x), & \text{if} \quad \alpha < 0 \\ {}_aI_x^{n-\alpha}f^{(n)}(x), & \text{if} \quad \alpha \geq 0 \end{cases}, \tag{4} $$

where $n = \lceil \alpha \rceil$ and ${}_aI_x^0 f^{(n)}(x) := f^{(n)}(x)$. Furthermore, if the function f fulfills that $f^{(k)}(a) = 0 \ \forall k \in \{0, 1, \cdots, n-1\}$, the Riemann−Liouville fractional derivative coincides with the Caputo fractional derivative, that is,

$$ {}_aD_x^\alpha f(x) = {}_a^CD_x^\alpha f(x). \tag{5} $$

So, applying the operator (3) with $a = 0$ to the function x^μ, with $\mu > -1$, we obtain the following result [7]:

$$ {}_0D_x^\alpha x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}x^{\mu-\alpha}, \quad \alpha \in \mathbb{R}\backslash\mathbb{Z}, \tag{6} $$

where if $1 \leq \lceil \alpha \rceil \leq \mu$ it is fulfilled that ${}_0D_x^\alpha x^\mu = {}_0^CD_x^\alpha x^\mu$.

2. Sets of fractional operators

Before continuing, it is necessary to mention that due to the large number of fractional operators that may exist [1–3, 8–23], some sets must be defined to fully characterize elements of fractional calculus. It is worth mentioning that characterizing elements of fractional calculus through sets is the main idea behind of the methodology known as fractional calculus of sets [24, 25]. So, considering a scalar function $h : \mathbb{R}^m \to \mathbb{R}$ and the canonical basis of $\mathbb{R}^m$ denoted by $\{\hat{e}_k\}_{k \geq 1}$, it is possible to define the following fractional operator of order α using Einstein notation

$$ o_x^\alpha h(x) := \hat{e}_k o_k^\alpha h(x). \tag{7} $$

Therefore, denoting by ∂_k^n the partial derivative of order n applied with respect to the k-th component of the vector x, using the previous operator it is possible to define the following set of fractional operators

$$ O_{x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \quad \text{and} \quad \lim_{\alpha \to n} o_k^\alpha h(x) = \partial_k^n h(x) \ \ \forall k \geq 1 \right\}, \tag{8} $$

which may be proved to be a nonempty set through the following set of fractional operators

$$O^n_{0,x,\alpha}(h) := \left\{ o^\alpha_x \;:\; \exists o^\alpha_k h(x) = \left(\partial^n_k + \mu(\alpha)\partial^\alpha_k\right)h(x) \quad \text{and} \quad \lim_{\alpha \to n} \mu(\alpha)\partial^\alpha_k h(x) = 0 \;\; \forall k \geq 1 \right\}, \tag{9}$$

with which it is possible to obtain the following result:

$$\textit{If} \quad o^\alpha_{i,x}, o^\alpha_{j,x} \in O^n_{x,\alpha}(h) \quad \text{with} \quad i \neq j \quad \Rightarrow \quad \exists o^\alpha_{k,x} = \frac{1}{2}\left(o^\alpha_{i,x} + o^\alpha_{j,x}\right) \in O^n_{x,\alpha}(h). \tag{10}$$

So, the complement of the set (8) may be defined as follows

$$O^{n,c}_{x,\alpha}(h) := \left\{ o^\alpha_x \;:\; \exists o^\alpha_k h(x) \quad \forall k \geq 1 \quad \text{and} \quad \lim_{\alpha \to n} o^\alpha_k h(x) \neq \partial^n_k h(x) \quad \text{in} \quad \text{at least one value } k \geq 1 \right\}, \tag{11}$$

with which it is possible to obtain the following result:

$$\textit{If} \quad o^\alpha_{i,x} = \hat{e}_k o^\alpha_{i,k} \in O^n_{x,\alpha}(h) \quad \Rightarrow \quad \exists o^\alpha_{j,x} = \hat{e}_k o^\alpha_{i,\sigma_j(k)} \in O^{n,c}_{x,\alpha}(h), \tag{12}$$

where $\sigma_j : \{1, 2, \cdots, m\} \to \{1, 2, \cdots, m\}$ denotes any permutation different from the identity. On the other hand, the set (8) may be considered as a generating set of sets of fractional tensor operators. For example, considering $\alpha, n \in \mathbb{R}^d$ with $\alpha = \hat{e}_k[\alpha]_k$ and $n = \hat{e}_k[n]_k$, it is possible to define the following set of fractional tensor operators

$$O^n_{x,\alpha}(h) := \left\{ o^\alpha_x \;:\; \exists o^\alpha_x h(x) \quad \text{and} \quad o^\alpha_x \in O^{[n]_1}_{x,[\alpha]_1}(h) \times O^{[n]_2}_{x,[\alpha]_2}(h) \times \cdots \times O^{[n]_d}_{x,[\alpha]_d}(h) \right\}. \tag{13}$$

3. Groups of fractional operators

Considering a function $h : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$, it is possible to define the following sets

$${}_m\mathrm{O}^n_{x,\alpha}(h) := \left\{ o^\alpha_x \;:\; o^\alpha_x \in \mathrm{O}^n_{x,\alpha}\left([h]_k\right) \;\; \forall k \leq m \right\}, \tag{14}$$

$${}_m\mathrm{O}^{n,c}_{x,\alpha}(h) := \left\{ o^\alpha_x \;:\; o^\alpha_x \in \mathrm{O}^{n,c}_{x,\alpha}\left([h]_k\right) \;\; \forall k \leq m \right\}, \tag{15}$$

$${}_m\mathrm{O}^{n,u}_{x,\alpha}(h) := {}_m\mathrm{O}^n_{x,\alpha}(h) \cup {}_m\mathrm{O}^{n,c}_{x,\alpha}(h), \tag{16}$$

where $[h]_k : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ denotes the k-th component of the function h. So, it is possible to define the following set of fractional operators

$${}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h) := \bigcap_{k \in \mathbb{Z}} m\mathrm{O}^{k,u}_{x,\alpha}(h), \tag{17}$$

which under the classical Hadamard product it is fulfilled that

$$o^0_x \circ h(x) := h(x) \quad \forall o^\alpha_x \in {}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h). \tag{18}$$

As a consequence, it is possible to define the following set of matrices

$$ {}_m\mathrm{M}_{x,\alpha}^{\infty}(h) := \left\{ A_{h,\alpha} = A_{h,\alpha}\left(o_x^\alpha\right) : o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h) \text{ and } A_{h,\alpha}(x) = \left([A_{h,\alpha}]_{jk}(x)\right) := \left(o_k^\alpha[h]_j(x)\right)\right\}, \tag{19} $$

and therefore, considering that when using the classical Hadamard product in general $o_x^{p\alpha} \circ o_x^{q\alpha} \neq o_x^{(p+q)\alpha}$. Assuming the existence of a fixed set of matrices ${}_m\mathrm{M}_{x,\alpha}^{\infty}(h)$, joined with a modified Hadamard product that fulfills the following property

$$ o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha} := \begin{cases} o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}, & \text{if } i \neq j \quad \text{(Hadamard product of type horizontal)} \\ o_{i,x}^{(p+q)\alpha}, & \text{if } i = j \quad \text{(Hadamard product of type vertical)} \end{cases}, \tag{20} $$

by omitting the function h, the resulting set ${}_m\mathrm{M}_{x,\alpha}^{\infty}(\cdot)$ has the ability to generate a group of fractional matrix operators A_α that fulfill the following equation

$$ A_\alpha\left(o_{i,x}^{p\alpha}\right) \circ A_\alpha\left(o_{j,x}^{q\alpha}\right) := \begin{cases} A_\alpha\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right), & \text{if } i \neq j \\ A_\alpha\left(o_{i,x}^{(p+q)\alpha}\right), & \text{if } i = j \end{cases}, \tag{21} $$

through the following set [24, 26]:

$$ {}_m\mathrm{G}_{FIM}(\alpha) := \left\{ A_\alpha^{\circ r} = A_\alpha\left(o_x^{r\alpha}\right) \ : \ \exists A_\alpha^{\circ r} \in {}_m\mathrm{M}_{x,\alpha}^{\infty}(\cdot) \ \forall r \in \mathbb{Z} \quad \text{and} \quad A_\alpha^{\circ r} = \left(\left[A_\alpha^{\circ r}\right]_{jk}\right) := \left(o_k^{r\alpha}\right)\right\}. \tag{22} $$

Where $\forall A_{i,\alpha}^{\circ p}, A_{j,\alpha}^{\circ q}, A_{j,\alpha}^{\circ r} \in {}_m\mathrm{G}_{FNR}(\alpha)$, with $i \neq j$, the following property is defined

$$ \left(A_{i,\alpha}^{\circ p} \circ A_{j,\alpha}^{\circ q}\right) \circ A_{j,\alpha}^{\circ r} = A_{i,\alpha}^{\circ p} \circ \left(A_{j,\alpha}^{\circ q} \circ A_{j,\alpha}^{\circ r}\right) = A_{k,\alpha}^{\circ 1} := A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{(q+r)\alpha}\right), \quad p,q,r \in \mathbb{Z}\setminus\{0\}, \tag{23} $$

since it is considered that through combinations of the Hadamard product of type horizontal and vertical the fractional operators are reduced to their minimal expression. As a consequence, it is fulfilled that

$$ \begin{aligned} \forall A_{k,\alpha}^{\circ 1} \in {}_m\mathrm{G}_{FIM}(\alpha) \quad \text{such that} \quad A_{k,\alpha}\left(o_{k,x}^{\alpha}\right) = A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right) \exists A_{k,\alpha}^{\circ r} = A_{k,\alpha}^{\circ (r-1)} \circ A_{k,\alpha}^{\circ 1} \\ = A_{k,\alpha}\left(o_{i,x}^{rp\alpha} \circ o_{j,x}^{rq\alpha}\right). \end{aligned} \tag{24} $$

It is necessary to mention that for each operator $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ it is possible to define a group [26], which is isomorphic to the group of integers under the addition, as shown by the following theorems:

Theorem 1.1 Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$. So, considering the modified Hadamard product given by (20), it is possible to define the following set of fractional matrix operators

$$ {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) := \left\{ A_\alpha^{\circ r} = A_\alpha\left(o_x^{r\alpha}\right) \ : \ r \in \mathbb{Z} \quad \text{and} \quad A_\alpha^{\circ r} = \left(\left[A_\alpha^{\circ r}\right]_{jk}\right) := \left(o_k^{r\alpha}\right) \right\}, \tag{25} $$

which corresponds to the Abelian group generated by the operator $A_\alpha\left(o_x^\alpha\right)$.

Proof: It should be noted that due to the way the set (25) is defined, just the Hadamard product of type vertical is applied among its elements. So, $\forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m G\left(A_\alpha\left(o_x^\alpha\right)\right)$ it is fulfilled that

$$ A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = \left(\left[A_\alpha^{\circ p}\right]_{jk}\right) \circ \left(\left[A_\alpha^{\circ q}\right]_{jk}\right) = \left(o_k^{(p+q)\alpha}\right) = \left(\left[A_\alpha^{\circ (p+q)}\right]_{jk}\right) = A_\alpha^{\circ (p+q)}, \tag{26} $$

with which it is possible to prove that the set ${}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right)$ fulfills the following properties, which correspond to the properties of an Abelian group:

$$ \begin{cases} \forall A_\alpha^{\circ p}, A_\alpha^{\circ q}, A_\alpha^{\circ r} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) \text{ it is fulfilled that } \left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) \circ A_\alpha^{\circ r} = A_\alpha^{\circ p} \circ \left(A_\alpha^{\circ q} \circ A_\alpha^{\circ r}\right) 0.1cm \\ \exists A_\alpha^{\circ 0} \in {}_m G\left(A_\alpha\left(o_x^\alpha\right)\right) \text{ such that } \forall A_\alpha^{\circ p} \in {}_m G\left(A_\alpha\left(o_x^\alpha\right)\right) \text{ it is fulfilled that } A_\alpha^{\circ 0} \circ A_\alpha^{\circ p} = A_\alpha^{\circ p} 0.1cm \\ \forall A_\alpha^{\circ p} \in {}_m G\left(A_\alpha\left(o_x^\alpha\right)\right) \ \exists A_\alpha^{\circ -p} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) \text{ such that } A_\alpha^{\circ p} \circ A_\alpha^{\circ -p} = A_\alpha^{\circ 0} 0.1cm \\ \forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) \text{ it is fulfilled that } A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = A_\alpha^{\circ q} \circ A_\alpha^{\circ p} \end{cases} .. \tag{27} $$

Theorem 1.2 Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ and let $(\mathbb{Z}, +)$ be the group of integers under the addition. So, the group generated by the operator $A_\alpha\left(o_x^\alpha\right)$ is isomorphic to the group $(\mathbb{Z}, +)$, that is,

$$ {}_m G\left(A_\alpha\left(o_x^\alpha\right)\right) \cong (\mathbb{Z}, +). \tag{28} $$

Proof: To prove the theorem it is enough to define a bijective homomorphism between the sets ${}_m G\left(A_\alpha\left(o_x^\alpha\right)\right)$ and $(\mathbb{Z}, +)$. Let $\psi : {}_m G\left(A_\alpha\left(o_x^\alpha\right)\right) \to (\mathbb{Z}, +)$ be a function with inverse function $\psi^{-1} : (\mathbb{Z}, +) \to {}_m G\left(A_\alpha\left(o_x^\alpha\right)\right)$. So, the functions ψ and ψ^{-1} may be defined as follows

$$ \psi\left(A_\alpha^{\circ r}\right) = r \ \text{ and } \ \psi^{-1}(r) = A_\alpha^{\circ r}, \tag{29} $$

with which it is possible to obtain the following results:

$$ \begin{cases} \forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m G\left(A_\alpha\left(o_x^\alpha\right)\right) \text{ it is fulfilled that } \psi\left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) = \psi\left(A_\alpha^{\circ (p+q)}\right) = p + q = \psi\left(A_\alpha^{\circ p}\right) + \psi\left(A_\alpha^{\circ q}\right) \\ \forall p, q \in (\mathbb{Z}, \ +) \text{ it is fulfilled that } \psi^{-1}(p+q) = A_\alpha^{\circ (p+q)} = A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = \psi^{-1}(p) \circ \psi^{-1}(q) \end{cases} . \tag{30} $$

Therefore, from the previous results, it follows that the function ψ defines an isomorphism between the sets ${}_m G\left(A_\alpha\left(o_x^\alpha\right)\right)$ and $(\mathbb{Z}, +)$.

Then, from the previous theorems it is possible to obtain the following corollaries:

Corollary 1.3 *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ and let $(\mathbb{Z}, +)$ be the group of integers under the addition. So, considering the modified Hadamard product given by (20) and some subgroup $\mathbb{H}$ of the group $(\mathbb{Z}, +)$, it is possible to define the following set of fractional matrix operators*

$$ {}_mG\left(A_\alpha\left(o_x^\alpha\right), \mathbb{H}\right) := \left\{ A_\alpha^{\circ r} = A_\alpha\left(o_x^{r\alpha}\right) \ : \ r \in \mathbb{H} \quad \text{and} \quad A_\alpha^{\circ r} = \left(\left[A_\alpha^{\circ r}\right]_{jk}\right) := \left(o_k^{r\alpha}\right) \right\}, \tag{31} $$

which corresponds to a subgroup of the group generated by the operator $A_\alpha\left(o_x^\alpha\right)$, that is,

$$ {}_mG\left(A_\alpha\left(o_x^\alpha\right), \mathbb{H}\right) \le {}_mG\left(A_\alpha\left(o_x^\alpha\right)\right). \tag{32} $$

Example 1 *Let $\mathbb{Z}_n$ be the set of residual classes less than n. So, considering a fractional operator $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ and the set $\mathbb{Z}_{14}$, it is possible to define the Abelian group of fractional matrix operators ${}_mG\left(A_\alpha\left(o_x^\alpha\right), \mathbb{Z}_{14}\right)$. Furthermore, all possible combinations of the elements of the group under the modified Hadamard product given by (20) are summarized below:*

$\circ$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$
$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$
$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$
$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$
$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$
$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$
$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$
$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$
$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$
$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$
$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$
$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$
$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$
$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$
$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$

Corollary 1.4 *Let $h : \mathbb{R}^m \to \mathbb{R}^m$ be a function such that $\exists_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$. So, if it is fulfilled the following condition*

$$ \forall o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h) \quad \exists_m G\left(A_\alpha\left(o_x^\alpha\right)\right) \subset {}_mG_{FIM}(\alpha), \tag{33} $$

such that ${}_mG\left(A_\alpha\left(o_x^\alpha\right)\right)$ is the group generated by the operator $A_\alpha\left(o_x^\alpha\right)$. As a consequence, it is fulfilled that

$$ {}_mG_{FIM}(\alpha) = \bigcup_{o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)} {}_mG\left(A_\alpha\left(o_x^\alpha\right)\right). \tag{34} $$

It is necessary to mention that the Corollary 1.3 allows generating groups of fractional operators under other operations, as shown in the following corollary:

Corollary 1.5 *Let $\mathbb{Z}_p^+$ be the set of positive residual classes less than p, with p a prime number. So, for each fractional operator $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$, it is possible to define the following set of fractional matrix operators*

$$ {}_mG^*\left(A_\alpha\left(o_x^\alpha\right), \mathbb{Z}_p^+\right) := \left\{A_\alpha^{\circ r} = A_\alpha\left(o_x^{r\alpha}\right) \;:\; r \in \mathbb{Z}_p^+ \quad \text{and} \quad A_\alpha^{\circ r} = \left(\left[A_\alpha^{\circ r}\right]_{jk}\right) := \left(o_k^{r\alpha}\right)\right\}, \tag{35} $$

which corresponds to an Abelian group under the following operation

$$ A_\alpha^{\circ r} * A_\alpha^{\circ s} = A_\alpha^{\circ rs}. \tag{36} $$

Example 2 *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_mMO_{x,\alpha}^{\infty,u}(h)$. So, considering the set $\mathbb{Z}_{13}^+$, it is possible to define the Abelian group of fractional matrix operators ${}_mG^*\left(A_\alpha\left(o_x^\alpha\right), \mathbb{Z}_{13}^+\right)$. Furthermore, all possible combinations of the elements of the group under the operation (36) are summarized below:*

$*$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$
$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$
$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 11}$
$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 10}$
$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 9}$
$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 8}$
$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 7}$
$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 6}$
$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 5}$
$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 4}$
$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 3}$
$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 2}$
$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 1}$

On the other hand, defining $A_\alpha(h) = \left(\left[A_\alpha(h)\right]_{jk}\right) := \left([h]_k\right)$, it is possible to obtain the following result:

$$ \forall A_\alpha^{\circ r} \in {}_mG_{FIM}(\alpha) \quad \exists A_{h,r\alpha} \in {}_m\mathrm{M}_{x,\alpha}^{\infty}(h) \quad \text{such that} \quad A_{h,r\alpha} := A_\alpha\left(o_x^{r\alpha}\right) \circ A_\alpha^T(h). \tag{37} $$

Therefore, if Φ_{FIM} denotes the iteration function of some fractional iterative method [26], it is possible to obtain the following results:

$$ \begin{aligned} \text{Let} \;\; \alpha_0 \in \mathbb{R}\backslash\mathbb{Z} \Rightarrow \;\; & \forall A_{h,\alpha_0} \in {}_m\mathrm{M}_{x,\alpha}^{\infty}(h) \exists \Phi_{FIM} \\ & = \Phi_{FIM}(A_{h,\alpha_0}) .\!\because\! \forall A_{h,\alpha_0} \quad \exists\{\Phi_{FIM}(A_{h,\alpha}) : \;\; \alpha \in \mathbb{R}\backslash\mathbb{Z}\}, \end{aligned} \tag{38} $$

$$\text{Let } \alpha_0 \in \mathbb{R}\backslash\mathbb{Z} \Rightarrow \quad \forall A_{\alpha_0}^{\circ 1} \in {}_m G_{FIM}(\alpha) \exists \Phi_{FIM} \\ = \Phi_{FIM}(A_{\alpha_0}) \therefore \forall A_{\alpha_0} \quad \exists \{\Phi_{FIM}(A_\alpha) : \quad \alpha \in \mathbb{R}\backslash\mathbb{Z}\}. \tag{39}$$

To finish this section, it is necessary to mention that the applications of fractional operators have spread to different fields of science such as finance [27, 28], economics [29], number theory through the Riemann zeta function [30, 31], and in engineering with the study for the manufacture of hybrid solar receivers [32, 33]. It is worth mentioning that there exists also a growing interest in fractional operators and their properties for solving nonlinear algebraic systems [24, 34–41], which is a classical problem in mathematics, physics and engineering, which consists of finding the set of zeros of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, that is,

$$\{\xi \in \Omega \quad : \quad \|f(\xi)\| = 0\}, \tag{40}$$

where $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ denotes any vector norm, or equivalently

$$\left\{\xi \in \Omega \quad : \quad [f]_k(\xi) = 0 \quad \forall k \geq 1\right\}. \tag{41}$$

Although finding the zeros of a function may seem like a simple problem, it is generally necessary to use numerical methods of the iterative type to solve it.

4. Fixed-point method

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a function. It is possible to build a sequence $\{x_i\}_{i \geq 1}$ by defining the following iterative method

$$x_{i+1} \coloneqq \Phi(x_i), \quad i = 0,1,2,\cdots. \tag{42}$$

So, if it is fulfilled that $x_i \to \xi \in \mathbb{R}^n$ and the function Φ is continuous around ξ, we obtain that

$$\xi = \lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} \Phi(x_i) = \Phi\left(\lim_{i\to\infty} x_i\right) = \Phi(\xi), \tag{43}$$

the above result is the reason by which the method (42) is known as the fixed-point method. Furthermore, the function Φ is called an iteration function. On the other hand, considering the following set

$$B(\xi;\delta) \coloneqq \{x \quad : \quad \|x - \xi\| < \delta\}, \tag{44}$$

it is possible to define the following corollary, which allows characterizing the order of convergence of an iteration function Φ through its Jacobian matrix $\Phi^{(1)}$ [7]:

Corollary 1.6 *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function. If Φ defines a sequence $\{x_i\}_{i \geq 1}$ such that $x_i \to \xi \in \mathbb{R}^n$. So, Φ has an order of convergence of order (at least) p in $B(\xi;\delta)$, where it is fulfilled that:*

$$p \coloneqq \begin{cases} 1, & \text{if } \lim_{x\to\xi} \left\|\Phi^{(1)}(x)\right\| \neq 0 \\ 2, & \text{if } \lim_{x\to\xi} \left\|\Phi^{(1)}(x)\right\| = 0 \end{cases}. \tag{45}$$

5. Fractional fixed-point method

Let $\mathbb{N}_0$ be the set $\mathbb{N}\cup\{0\}$, if $\gamma\in\mathbb{N}_0^m$ and $x\in\mathbb{R}^m$, then it is possible to define the following multi-index notation

$$\begin{cases} \gamma! := \prod_{k=1}^{m} [\gamma]_k!, \quad |\gamma| := \sum_{k=1}^{m} [\gamma]_k 0.1cm, \quad x^\gamma := \prod_{k=1}^{m} [x]_k^{[\gamma]_k} \\ \dfrac{\partial^\gamma}{\partial x^\gamma} := \dfrac{\partial^{[\gamma]_1}}{\partial [x]_1^{[\gamma]_1}} \dfrac{\partial^{[\gamma]_2}}{\partial [x]_2^{[\gamma]_2}} \cdots \dfrac{\partial^{[\gamma]_m}}{\partial [x]_m^{[\gamma]_m}} \end{cases}. \tag{46}$$

So, considering a function $h:\Omega\subset\mathbb{R}^m\to\mathbb{R}$ and the fractional operator

$$s_x^{\alpha\gamma}\left(o_x^\alpha\right) := o_1^{\alpha[\gamma]_1} o_2^{\alpha[\gamma]_2} \cdots o_m^{\alpha[\gamma]_m}, \tag{47}$$

it is possible to define the following set of fractional operators

$$S_{x,\alpha}^{n,\gamma}(h) := \left\{ s_x^{\alpha\gamma} = s_x^{\alpha\gamma}\left(o_x^\alpha\right) : \exists s_x^{\alpha\gamma} h(x) \text{ with } o_x^\alpha \in O_{x,\alpha}^s(h)\ \forall s \le n^2 \right. \\ \left. \text{and } \lim_{\alpha\to k} s_x^{\alpha\gamma} h(x) = \frac{\partial^{k\gamma}}{\partial x^{k\gamma}} h(x)\ \forall \alpha, |\gamma| \le n \right\}, \tag{48}$$

from which it is possible to obtain the following results:

$$\text{If } s_x^{\alpha\gamma} \in S_{x,\alpha}^{n,\gamma}(h) \Rightarrow \begin{cases} \lim_{\alpha\to 0} s_x^{\alpha\gamma} h(x) = o_1^0 o_2^0 \cdots o_m^0 h(x) = h(x) \\ \lim_{\alpha\to 1} s_x^{\alpha\gamma} h(x) = o_1^{[\gamma]_1} o_2^{[\gamma]_2} \cdots o_m^{[\gamma]_m} h(x) = \dfrac{\partial^\gamma}{\partial x^\gamma} h(x)\ \forall |\gamma| \le n \\ \lim_{\alpha\to q} s_x^{\alpha\gamma} h(x) = o_1^{q[\gamma]_1} o_2^{q[\gamma]_2} \cdots o_m^{q[\gamma]_m} h(x) = \dfrac{\partial^{q\gamma}}{\partial x^{q\gamma}} h(x)\ \forall q|\gamma| \le qn \\ \lim_{\alpha\to n} s_x^{\alpha\gamma} h(x) = o_1^{n[\gamma]_1} o_2^{n[\gamma]_2} \cdots o_m^{n[\gamma]_m} h(x) = \dfrac{\partial^{n\gamma}}{\partial x^{n\gamma}} h(x)\ \forall n|\gamma| \le n^2 \end{cases}, \tag{49}$$

and as a consequence, considering a function $h:\Omega\subset\mathbb{R}^m\to\mathbb{R}^m$, it is possible to define the following set of fractional operators

$$_m S_{x,\alpha}^{n,\gamma}(h) := \left\{ s_x^{\alpha\gamma} \ : \ s_x^{\alpha\gamma} \in S_{x,\alpha}^{n,\gamma}\left([h]_k\right)\ \forall k \le m \right\}. \tag{50}$$

On the other hand, using little-o notation it is possible to obtain the following result:

$$\text{If } x\in B(a;\delta) \Rightarrow \lim_{x\to a} \frac{o((x-a)^\gamma)}{(x-a)^\gamma} \to 0 \quad \forall |\gamma| \ge 1, \tag{51}$$

with which it is possible to define the following set of functions

$$R_{\alpha\gamma}^n(a) := \left\{ r_{\alpha\gamma}^n \ : \ \lim_{x\to a} \left\| r_{\alpha\gamma}^n(x) \right\| = 0\ \forall |\gamma| \ge n \text{ and } \left\| r_{\alpha\gamma}^n(x) \right\| \le o(\|x-a\|^n)\ \forall x\in B(a;\delta) \right\}, \tag{52}$$

where $r_{\alpha\gamma}^{n} : B(a;\delta) \subset \Omega \to \mathbb{R}^m$. So, considering the previous set and some $B(a;\delta) \subset \Omega$, it is possible to define the following sets of fractional operators

$$m T_{x,\alpha,p}^{n,q,\gamma}(a,h) := \left\{ t_x^{\alpha,p} = t_x^{\alpha,p}\left(s_x^{\alpha\gamma}\right) : \ s_x^{\alpha\gamma} \in {}_m S_{x,\alpha}^{M,\gamma}(h) \quad \text{and} \right.$$

$$\left. t_x^{\alpha,p} h(x) := \sum_{|\gamma|=0}^{p} \frac{1}{\gamma!} \hat{e}_j s_x^{\alpha\gamma}[h]_j(a)(x-a)^\gamma + r_{\alpha\gamma}^{p}(x) \begin{array}{l} \forall \alpha \le n \\ \forall p \le q \end{array} \right\}, \tag{53}$$

$$m T_{x,\alpha}^{\infty,\gamma}(a,h) := \left\{ t_x^{\alpha,\infty} = t_x^{\alpha,\infty}\left(s_x^{\alpha\gamma}\right) : \ s_x^{\alpha\gamma} \in {}_m S_{x,\alpha}^{\infty,\gamma}(h) \quad \text{and} \right.$$

$$\left. t_x^{\alpha,\infty} h(x) := \sum_{|\gamma|=0}^{\infty} \frac{1}{\gamma!} \hat{e}_j s_x^{\alpha\gamma}[h]_j(a)(x-a)^\gamma \right\}, \tag{54}$$

which allow generalizing the Taylor series expansion of a vector-valued function in multi-index notation [7], where $M = \max\{n, q\}$. As a consequence, it is possible to obtain the following results:

$$\text{If} \ \ t_x^{\alpha,p} \in {}_m T_{x,\alpha,p}^{1,q,\gamma}(a,h) \ \ \text{and} \ \ \alpha \to 1 \ \ \Rightarrow$$
$$t_x^{1,p} h(x) = h(a) + \sum_{|\gamma|=1}^{p} \frac{1}{\gamma!} \hat{e}_j \frac{\partial^\gamma}{\partial x^\gamma}[h]_j(a)(x-a)^\gamma + r_\gamma^p(x), \tag{55}$$

$$\text{If} \ \ t_x^{\alpha,p} \in {}_m T_{x,\alpha,p}^{n,1,\gamma}(a,h) \ \ \text{and} \ \ p \to 1 \ \ \Rightarrow \ \ t_x^{\alpha,1} h(x) = h(a) + \sum_{k=1}^{m} \hat{e}_j o_k^\alpha[h]_j(a)[(x-a)]_k + r_{\alpha\gamma}^1(x). \tag{56}$$

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$. So, for some $x_i \in B(\xi;\delta) \subset \Omega$ and for some fractional operator $t_x^{\alpha,\infty} \in {}_n \mathrm{T}_{x,\alpha}^{\infty,\gamma}(x_i, f)$, it is possible to define a type of linear approximation of the function f around the value x_i as follows

$$t_x^{\alpha,\infty} f(x) \approx f(x_i) + \sum_{k=1}^{n} \hat{e}_j o_k^\alpha[f]_j(x_i)[(x-x_i)]_k, \tag{57}$$

which may be rewritten more compactly as follows

$$t_x^{\alpha,\infty} f(x) \approx f(x_i) + \left(o_k^\alpha[f]_j(x_i)\right)(x - x_i). \tag{58}$$

where $\left(o_k^\alpha[f]_j(x_i)\right)$ denotes a square matrix. On the other hand, if $x \to \xi$ and since $\|f(\xi)\| = 0$, it follows that

$$0 \approx f(x_i) + \left(o_k^\alpha[f]_j(x_i)\right)(\xi - x_i) \quad \Rightarrow \quad \xi \approx x_i - \left(o_k^\alpha[f]_j(x_i)\right)^{-1} f(x_i). \tag{59}$$

So, defining the following matrix

$$A_{f,\alpha}(x) = \left(\left[A_{f,\alpha}\right]_{jk}(x)\right) := \left(o_k^\alpha[f]_j(x)\right)^{-1}, \tag{60}$$

it is possible to define the following fractional iterative method

$$x_{i+1} := \Phi(\alpha, x_i) = x_i - A_{f,\alpha}(x_i)f(x_i), \quad i = 0,1,2,\cdots, \tag{61}$$

which corresponds to the more general case of the fractional Newton-Raphson method [25, 36].

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$. So, considering an iteration function $\Phi : (\mathbb{R}\backslash\mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$, the iteration function of a fractional iterative method may be written in general form as follows

$$\Phi(\alpha, x) := x - A_{g,\alpha}(x)f(x), \quad \alpha \in \mathbb{R}\backslash\mathbb{Z}, \tag{62}$$

where $A_{g,\alpha}$ is a matrix that depends, in at least one of its entries, on fractional operators of order α applied to some function $g : \mathbb{R}^n \to \mathbb{R}^n$, whose particular case occurs when $g = f$. So, it is possible to define in a general way a fractional fixed-point method as follows

$$x_{i+1} := \Phi(\alpha, x_i), \quad i = 0,1,2,\cdots. \tag{63}$$

Before continuing, it is worth mentioning that one of the main advantages of fractional iterative methods is that the initial condition x_0 can remain fixed, with which it is enough to vary the order α of the fractional operators involved until generating a sequence convergent $\{x_i\}_{i\geq 1}$ to the value $\xi \in \Omega$. Since the order α of the fractional operators is varied, different values of α can generate different convergent sequences to the same value ξ but with a different number of iterations. So, it is possible to define the following set

$$\mathrm{Conv}_\delta(\xi) := \left\{ \Phi : \lim_{x \to \xi} \Phi(\alpha,\ x) = \xi_\alpha \in B(\xi;\delta) \right\} \tag{64}$$

which may be interpreted as the set of fractional fixed-point methods that define a convergent sequence $\{x_i\}_{i\geq 1}$ to some value $\xi_\alpha \in B(\xi;\delta)$. So, denoting by $\mathrm{card}(\cdot)$ the cardinality of a set, under certain conditions it is possible to prove the following result (see reference [24], proof of Theorem 2):

$$\mathrm{card}(\mathrm{Conv}_\delta(\xi)) = \mathrm{card}(\mathbb{R}), \tag{65}$$

from which it follows that the set (64) is generated by an uncountable family of fractional fixed-point methods. Before continuing, it is necessary to define the following proposition [7]:

Proposition 1.7 *Let* $\Phi : (\mathbb{R}\backslash\mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$ *be an iteration function such that* $\Phi \in \mathrm{Conv}_\delta(\xi)$ *in a region* Ω*. So, if* Φ *is given by the equation (62) and fulfills the following condition*

$$\lim_{x \to \xi} A_{g,\alpha}(x) = \left(f^{(1)}(\xi)\right)^{-1}. \tag{66}$$

Then, Φ fulfills a necessary (but not sufficient) condition to be convergent of order (at least) quadratic in $B(\xi;\delta)$.

Proof: If Φ is given by the equation (62), the k-th component of the function Φ may be written as follows

$$[\Phi]_k(\alpha, x) = [x]_k - \sum_{j=1}^{n} [A_{g,\alpha}]kj(x)[f]_j(x), \tag{67}$$

and considering that $f^{(1)}(x) = \left(\left[f^{(1)}\right]jl(x)\right) := (\partial l[f]j(x))$, it is possible to obtain the following result

$$\left[\Phi^{(1)}\right]_{kl}(\alpha, x) = \partial_l[\Phi]_k(\alpha, x) = \delta_{kl} - \sum_{j=1}^{n}\left(\left[A_{g,\alpha}\right]_{kj}(x)\left[f^{(1)}\right]_{jl}(x) + \left(\partial_l\left[A_{g,\alpha}\right]_{kj}(x)\right)[f]_j(x)\right), \tag{68}$$

where δ_{kl} denotes the Kronecker delta. On the other hand, since f has a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$, it follows that

$$\left[\Phi^{(1)}\right]_{kl}(\alpha, \xi) = \delta_{kl} - \sum_{j=1}^{n}\left[A_{g,a}\right]_{kj}(\xi)\left[f^{(1)}\right]_{jl}(\xi). \tag{69}$$

Then, if $\Phi \in \mathrm{Conv}_\delta(\xi)$ and has an order of convergence (at least) quadratic in $B(\xi;\delta)$, by the Corollary 1.6, it is fulfilled the following condition

$$\sum_{j=1}^{n}\left[A_{g,a}\right]_{kj}(\xi)\left[f^{(1)}\right]_{jl}(\xi) = \delta_{kl}, \quad \forall k,l \le n, \tag{70}$$

which may be rewritten more compactly as follows

$$A_{g,a}(\xi)f^{(1)}(\xi) = I_n, \tag{71}$$

where I_n denotes the identity matrix of $n \times n$. Therefore, any matrix $A_{g,a}$ that fulfills the following condition

$$\lim_{x\to\xi} A_{g,a}(x) = \left(f^{(1)}(\xi)\right)^{-1}, \tag{72}$$

ensures that the iteration function Φ given by the equation (62), fulfills a necessary (but not sufficient) condition to be convergent of order (at least) quadratic in $B(\xi;\delta)$.

Considering the Corollary 1.6 and the Proposition 1.7, it is possible to define the following sets to classify the order of convergence of some fractional iterative methods:

$$\mathrm{Ord}^1(\xi) := \left\{\Phi \in \mathrm{Conv}_\delta(\xi) : \lim_{x\to\xi}\left\|\Phi^{(1)}(a, \ x)\right\| \neq 0\right\}, \tag{73}$$

$$\mathrm{Ord}^2(\xi) := \left\{\Phi \in \mathrm{Conv}_\delta(\xi) : \lim_{x\to\xi}\left\|\Phi^{(1)}(a, \ x)\right\| = 0\right\}, \tag{74}$$

$$\mathrm{ord}^1(\xi) := \left\{\Phi \in \mathrm{Conv}_\delta(\xi) : \lim_{x\to\xi} A_{g,a}(x) \neq \left(f^{(1)}(\xi)\right)^{-1} \text{ or } \lim_{\alpha\to 1} A_{g,a}(\xi) \neq \left(f^{(1)}(\xi)\right)^{-1}\right\}, \tag{75}$$

$$\mathrm{ord}^2(\xi) := \left\{\Phi \in \mathrm{Conv}_\delta(\xi) : \lim_{x\to\xi} A_{g,a}(x) \neq \left(f^{(1)}(\xi)\right)^{-1} \text{ or } \lim_{\alpha\to 1} A_{g,a}(\xi) \neq \left(f^{(1)}(\xi)\right)^{-1}\right\}, \tag{76}$$

On the other hand, considering that depending on the nature of the function f, there exist cases in which the Newton-Raphson method can present an order of convergence (at least) linear [7]. So, it is possible to obtain the following relations between the previous sets

$$\mathrm{ord}^1(\xi) \subset \mathrm{Ord}^1(\xi) \quad \text{and} \quad \mathrm{ord}^2(\xi) \subset \mathrm{Ord}^1(\xi) \cup \mathrm{Ord}^2(\xi), \tag{77}$$

with which it is possible to define the following sets

$$\mathrm{Ord}_2^1(\xi) := \mathrm{ord}^2(\xi) \cap \mathrm{Ord}^1(\xi) \quad \text{and} \quad \mathrm{Ord}_2^2(\xi) := \mathrm{ord}^2(\xi) \cap \mathrm{Ord}^2(\xi). \tag{78}$$

5.1 Acceleration of the order of convergence of the set $\mathrm{Ord}_2^1(\xi)$

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$, and denoting by Φ_{NR} to the iteration function of the Newton-Raphson method, it is possible to define the following set of functions

$$\mathrm{Ord}_{NR}^2(\xi) := \left\{ f : \lim_{x \to \xi} \left\| \Phi_{NR}^{(1)}(x) \right\| = 0 \right\}. \tag{79}$$

So, it is possible to define the following corollary:

Corollary 1.8 *Let* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a function such that* $f \in \mathrm{Ord}_{NR}^2(\xi)$, *and let* $\Phi : (\mathbb{R}\backslash\mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$ *be an iteration function given by the equation (62) such that* $\Phi \in \mathrm{ord}^1(\xi)$. *So, if* Φ *also fulfills the following condition*

$$\lim_{\alpha \to 1} A_{g,\alpha}(\xi) = \left(f^{(1)}(\xi) \right)^{-1}. \tag{80}$$

Then, $\Phi \in \mathrm{Ord}_2^1(\xi)$. Therefore, it is possible to assign a positive value δ_0, and replace the order α of the fractional operators of the matrix $A_{g,\alpha}$ by the following function

$$\alpha_f\left([x]_k, x\right) := \begin{cases} \alpha, & \text{if} |[x]_k| \neq 0 \quad \text{and} \quad \|f(x)\| \geq \delta_0 \\ 1 & \text{if} |[x]_k| = 0 \quad \text{or} \quad \|f(x)\| \geq \delta_0 \end{cases}, \tag{81}$$

obtaining a new matrix that may be denoted as follows

$$A_{g,\alpha_f}(x) = \left(\left[A_{g,\alpha_f} \right]_{jk}(x) \right), \quad \alpha \in \mathbb{R}\backslash\mathbb{Z}, \tag{82}$$

and with which it is fulfilled that $\Phi \in \mathrm{Ord}_2^2(\xi)$.

It is necessary to mention that, for practical purposes, it may be defined that if a fractional iterative method Φ fulfills the properties of the Corollary 1.8 and uses the function (81), it may be called a fractional iterative method accelerated. Finally, it is necessary to mention that fractional iterative methods may be defined in the complex space [24], that is,

$$\{ \Phi(\alpha, \ x) : \alpha \in \mathbb{R}\backslash\mathbb{Z} \quad \text{and} \quad x \in \mathbb{C}^n \}. \tag{83}$$

However, due to the part of the integral operator that fractional operators usually have, it may be considered that in the matrix $A_{g,\alpha}$ each fractional operator o_k^α is obtained for a real variable $[x]_k$, and if the result allows it, this variable is subsequently substituted by a complex variable $[x_i]k$, that is,

$$A_{g,\alpha}(x_i) := A_{g,\alpha}(x)\big|_{x\longrightarrow xi}, \quad x\in\mathbb{R}^n, \quad x_i\in\mathbb{C}^n. \tag{84}$$

Therefore, it is possible to obtain the following corollaries:

Corollary 1.9 *Let $f:\Omega\subset\mathbb{C}^n\to\mathbb{C}^n$ be a function such that $f\in\mathrm{Ord}^2_{NR}(\xi)$, let $g:\mathbb{C}^n\to\mathbb{C}^n$ be a function such that $g^{(1)}(x)=f^{(1)}(x)\ \forall x\in B(\xi;\delta)$, and let $\Phi:(\mathbb{R}\backslash\mathbb{Z})\times\mathbb{C}^n\to\mathbb{C}^n$ be an iteration function given by the equation (62). So, for each operator $o_x^\alpha\in{}_nO^1_{x,\alpha}(g)$ such that there exists the matrix $A^{-1}_{g,\alpha}=A_\alpha\left(o_x^\alpha\right)\circ A^T_\alpha(g)$, it follows that the matrix fulfills the following condition*

$$\lim_{\alpha\to1} A_{g,\alpha}(x)=\left(f^{(1)}(x)\right)^{-1}\ \forall x\in B(\xi;\delta). \tag{85}$$

As a consequence, by the Corollary 1.8, if $\Phi(A_{g,\alpha})\in Ord^1_2(\xi)\ \Rightarrow\ \Phi\left(A_{g,\alpha_f}\right)\in Ord^2_2(\xi)$.

Corollary 1.10 *Let $f:\Omega\subset\mathbb{C}^n\to\mathbb{C}^n$ be a function such that $f\in Ord^2_{NR}(\xi)$, let $\{g_k\}_{k=1}^N$ be a finite sequence of functions $g_k:\mathbb{C}^n\to\mathbb{C}^n$ such that it defines a finite sequence of operators $\left\{o^\alpha_{k,x}\right\}_{k=1}^N$ through the following condition*

$$o^\alpha_{k,x}\in{}_nMO^{\infty,u}_{x,\alpha}\left(g_k\right)\ \forall k\geq1, \tag{86}$$

and let $\Phi:(\mathbb{R}\backslash\mathbb{Z})\times\mathbb{C}^n\to\mathbb{C}^n$ be an iteration function given by the Eq. (62). So, if there exists a matrix $A_{N,\alpha}$ such that it fulfills the following conditions

$$\exists A^{-1}_{N,\alpha}=\sum_{k=1}^N A_\alpha\left(o^\alpha_{k,x}\right)\circ A^T_\alpha\left(g_k\right) \quad \text{and} \quad \lim_{\alpha\to1}A_{N,\alpha}(x)=\left(f^{(1)}(x)\right)^{-1}\ \forall x\in B(\xi;\delta). \tag{87}$$

As a consequence, by the Corollary 1.8, if $\Phi(A_{N,\alpha})\in\mathrm{Ord}^1_2(\xi)\ \Rightarrow\ \Phi\left(A_{N,\alpha_f}\right)\in\mathrm{Ord}^2_2(\xi)$.

6. Conclusions

It is worth mentioning that it is feasible to develop more complex algebraic structures of fractional operators using the presented results. For example, without loss of generality, considering the modified Hadamard product (20) and the operation (36), a commutative and unitary ring of fractional operators may be defined as follows

$${}_m\mathrm{R}\left(A_\alpha\left(o_x^\alpha\right)\right) := \left({}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right),\ \circ, *\right), \tag{88}$$

in which it is not difficult to verify the following properties:

1. The pair $\left({}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right),\ \circ\right)$ is an Abelian group.

2. The pair $\left({}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right),\ *\right)$ is an Abelian monoid.

3. $\forall A_\alpha^{\circ p}, A_\alpha^{\circ q}, A_\alpha^{\circ r} \in {}_m\mathrm{R}\left(A_\alpha\left(o_x^\alpha\right)\right)$, the operation $*$ is distributive with respect to the operation $\circ$, that is,

$$\begin{cases} A_\alpha^{\circ p} * \left(A_\alpha^{\circ q} \circ A_\alpha^{\circ r}\right) = \left(A_\alpha^{\circ p} * A_\alpha^{\circ q}\right) \circ \left(A_\alpha^{\circ p} * A_\alpha^{\circ r}\right) \\ \left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) * A_\alpha^{\circ r} = \left(A_\alpha^{\circ p} * A_\alpha^{\circ r}\right) \circ \left(A_\alpha^{\circ q} * A_\alpha^{\circ r}\right) \end{cases}. \tag{89}$$

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